The Sprague–Grundy function of the game Euclid

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Abstract

We show that the Sprague–Grundy function of the game *Euclid* is given by $g(x, y) = \lfloor |y/x - x/y| \rfloor$ for $x, y \geq 1$.

Keywords: Combinatorial game; Impartial game; Euclid; Sprague–Grundy function.

1 Introduction

Euclid is a two-player game based on the Euclidean algorithm for computing the greatest common divisor. The game was introduced by Cole and Davie [2].

A position in the game consists of a pair (x, y) of positive integers. Two players alternate moving; a move consists of subtracting from the larger integer any positive multiple of the smaller one, as long as the result is still positive. The game ends when no more moves are possible, i.e., when position (d, d) is reached, where $d = \gcd(x, y)$. The last player to move is the winner. This game is also discussed in [3, 4, 5, 6].

1.1 Impartial games and the Sprague–Grundy function

We briefly review the Sprague–Grundy theory of impartial games [1].

An impartial game can be represented by a directed acyclic graph $G = (V, E)$. Each position in the game corresponds to a vertex in G, and edges join vertices according to the game's legal moves. A token is initially placed on some vertex $v \in V$. Two players take turns moving the token from its current vertex to one of its followers. The player who moves the token into a sink wins.

Given two games G_1 and G_2, their *sum* $G_1 + G_2$ is played as follows: On each turn, a player chooses one of G_1, G_2, and moves on it, leaving the other game untouched. The game ends when no moves are possible on G_1 nor on G_2.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. Given a finite subset $S \subset \mathbb{N}$, let $\operatorname{mex} S = \min(\mathbb{N} \setminus S)$ denote the smallest natural number not in S. Then, given a game $G = (V, E)$, its *Sprague–Grundy function* (or just *Grundy function*) $g : V \to \mathbb{N}$ is defined recursively by

$$g(u) = \operatorname{mex}\big\{g(v) \bigm| (u, v) \in E\big\}, \qquad \text{for } u \in V. \tag{1}$$

This recursion starts by assigning sinks the value 0.

The Grundy function g satisfies the following two important properties:

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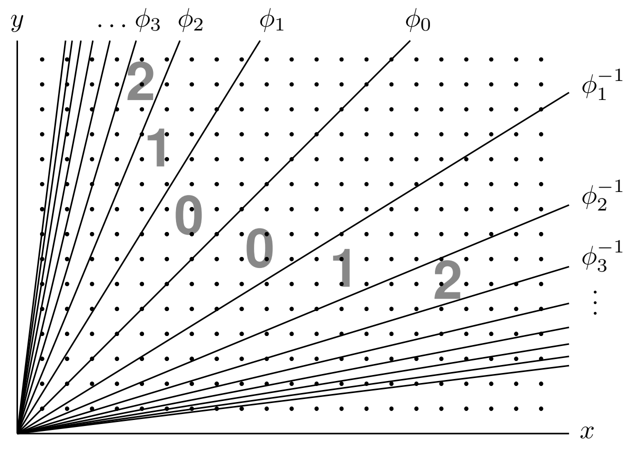

Figure 1: The Grundy function of the game Euclid. The rays' slopes are indicated.

1. If the two players play optimally, then the game starting from vertex $v \in V$ is a win for the second player if and only if $g(v) = 0$.

2. If $G = G_1 + G_2$ and $v = (v_1, v_2) \in V_1 \times V_2$, then $g(v)$ is the bitwise XOR, or *nim-sum*, of the binary representations of $g(v_1)$ and $g(v_2)$.

Clearly, the sum of games is an associative operation, as is the nim-sum operation. Therefore, knowledge of the Grundy function provides a winning strategy for the sum of any number of games.

2 Our result

It is known [2, 4, 5, 6] that a position (x, y), $x \leq y$ in Euclid is a first-player win if and only if $y > \phi x$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. We generalize this result by calculating the game's Grundy function. This allows us to play the sum of a game of Euclid with other impartial games.

Theorem 1 *The Grundy function of the game Euclid is given by*

$$g(x, y) = \left\lfloor \left| \frac{y}{x} - \frac{x}{y} \right| \right\rfloor. \tag{2}$$

For example, position $(5, 16)$ has followers $(5, 11)$, $(5, 6)$, and $(5, 1)$. According to formula (2), the Grundy values of the followers are 1, 0, and 4, respectively. And $g(5, 16) = 2$, obeying the *mex property* (1).

Proof Let us examine $g(x, y)$ for real $x, y > 0$. For every $n \in \mathbb{N}$, let ϕ_n be the positive real number satisfying

$$\phi_n - \phi_n^{-1} = n,$$

i.e., $\phi_n = (n + \sqrt{n^2 + 4})/2$. (Note that $\phi_1 = \phi$.) Then the points (x, y), $x \leq y$, with $g(x, y) = n$ are those lying between the rays $y = \phi_n x$ and $y = \phi_{n+1} x$ (see Figure 1).

Note that ϕ_n is irrational for $n \geq 1$, so the rays $y = \phi_n^{\pm 1} x$ do not pass through any lattice point. Therefore, we can ignore the value of g along those rays.

Given integers $x \geq 1$ and $n \geq 0$, the set of reals y for which $g(x, y) = n$ is the union of two intervals (disjoint for $n \geq 1$):

$$S_{x,n} = \big(\phi_{n+1}^{-1}x,\ \phi_n^{-1}x\big] \cup \big[\phi_n x,\ \phi_{n+1}x\big).$$

Now, the set

$$S'_{x,n} = \big[\phi_{n+1}^{-1}x,\ \phi_n^{-1}x\big) \cup \big[\phi_n x,\ \phi_{n+1}x\big)$$

(which differs from $S_{x,n}$ only at the endpoints) contains for every real residue r, $0 \leq r < x$, a unique y for which $y \bmod x = r$. This is because, first, the total length of $S'_{x,n}$ is

$$(\phi_{n+1} - \phi_{n+1}^{-1})x - (\phi_n - \phi_n^{-1})x = (n+1)x - nx = x,$$

and second, the inner endpoints $\phi_n^{-1}x$ and $\phi_n x$ are congruent modulo x.

Therefore, $S_{x,n}$ contains exactly x integers, and they are all distinct modulo x.

Now, suppose we are in position (x, y), $x \leq y$, with $g(x, y) = n$. For every $m < n$, all the positions (x, y') with $g(x, y') = m$ have $y' < y$; one of these positions must satisfy $y' \equiv y \pmod{x}$, so it is reachable from position (x, y).

On the other hand, we cannot move from (x, y) to another position (x, y') with $g(x, y') = n$, since all such positions have $y \not\equiv y' \pmod{x}$.

Therefore, formula (2) satisfies the mex property (1), so it is the Grundy function of the game Euclid, as claimed. ∎

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