Weak ϵ-nets and interval chains

Noga Alon* — Tel Aviv University — nogaa@post.tau.ac.il

Haim Kaplan† — Tel Aviv University — haimk@post.tau.ac.il

Gabriel Nivasch‡ — Tel Aviv University — gnivasch@post.tau.ac.il

Micha Sharir§ — Tel Aviv University — michas@post.tau.ac.il

Shakhar Smorodinsky¶ — Hebrew University of Jerusalem — shakhar@courant.nyu.edu

Abstract

We construct weak ϵ-nets of almost linear size for certain types of point sets. Specifically, for planar point sets in convex position we construct weak $\frac{1}{r}$-nets of size $O(r\alpha(r))$, where $\alpha(r)$ denotes the inverse Ackermann function. For point sets along the moment curve in $\mathbb{R}^d$ we construct weak $\frac{1}{r}$-nets of size $r \cdot 2^{\mathrm{poly}(\alpha(r))}$, where the degree of the polynomial in the exponent depends (quadratically) on d.

Our constructions result from a reduction to a new problem, which we call *stabbing interval chains with j-tuples*. Given the range of integers $N = [1, n]$, an interval chain of length k is a sequence of k consecutive, disjoint, nonempty intervals contained in N. A j-tuple $\overline{p} = (p_1, \ldots, p_j)$ is said to *stab* an interval chain $C = I_1 \cdots I_k$ if each p_i falls on a different interval of C. The problem is to construct a small-size family $\mathcal{Z}$ of j-tuples that stabs all k-interval chains in N.

Let $z_k^{(j)}(n)$ denote the minimum size of such a family $\mathcal{Z}$. We derive almost-tight upper and lower bounds for $z_k^{(j)}(n)$ for every fixed j; our bounds involve functions $\alpha_m(n)$ of the inverse Ackermann hierarchy. Specifically, we show that for $j = 3$ we have $z_k^{(3)}(n) = \Theta\big(n\alpha_{\lfloor k/2 \rfloor}(n)\big)$ for all $k \geq 6$. For each $j \geq 4$ we construct a pair of functions $P'_j(m)$, $Q'_j(m)$, almost equal asymptotically, such that $z^{(j)}_{P'_j(m)}(n) = O(n\alpha_m(n))$ and $z^{(j)}_{Q'_j(m)}(n) = \Omega(n\alpha_m(n))$.

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1 Introduction

Let S be an n-point set in $\mathbb{R}^d$, and let ϵ be a real number, $0 < \epsilon < 1$. A *weak ϵ-net* for S (with respect to convex sets) is a set of points $N \subset \mathbb{R}^d$, such that every convex set in $\mathbb{R}^d$ that contains at least ϵn points of S contains a point of N.[1] For convenience, we let $r = 1/\epsilon$, and we speak of weak $\frac{1}{r}$-nets, $r > 1$, so our bounds increase with r.

Alon et al. [2] showed that, for every d, for every finite $S \subset \mathbb{R}^d$ and every $r > 1$ there exists a weak $\frac{1}{r}$-net of size at most $f_d(r)$, for some family of functions f_d, each depending only on r.

The best known upper bound for the planar case is $f_2(r) = O(r^2)$, by Alon et al. [2] (see also Chazelle et al. [7]). For general $d \geq 3$ we have $f_d(r) = O\big(r^d(\log r)^{c(d)}\big)$, for some constants $c(d)$. This was first shown by Chazelle et al. [7], and later on by Matoušek and Wagner [12] via an alternative, simpler technique.

On the other hand, there are no known lower bounds for fixed d, besides the trivial $f_d(r) = \Omega(r)$. (Matoušek [10] showed, though, that $f_d(r)$ increases exponentially in d for fixed r; specifically, $f_d(50) = \Omega\Big(e^{\sqrt{d/2}}\Big)$.)

If the points of S lie in certain special configurations, better bounds exist on the size of the weak ϵ-net. For example, Chazelle et al. [7] showed that if $S \subset \mathbb{R}^2$ is in convex position, then S has a weak $\frac{1}{r}$-net of size $O\big(r(\log r)^{\log_2 3}\big) = O\big(r(\log r)^{1.59}\big)$. Furthermore, if S is the vertex set of a regular n-gon, then S admits a weak $\frac{1}{r}$-net of size $\Theta(r)$.

The techniques of Matoušek and Wagner [12] also yield improved bounds for some special cases. That is, they showed that if the points of $S \subset \mathbb{R}^d$ lie along the *moment curve*

$$\mu_d = \{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\}, \tag{1.1}$$

[1]The set N is called a *weak* ϵ-net because we do not necessarily have $N \subseteq S$; otherwise, N would be a regular (or "strong") ϵ-net. The need to consider *weak* ϵ-nets here stems from the fact that the system of all convex sets in $\mathbb{R}^d$ has *infinite VC-dimension*. For a full discussion, see Matoušek [11, Ch. 10].

then S has a weak $\frac{1}{r}$-net of size $O\big(r(\log r)^{c'(d)}\big)$, for some constants $c'(d) \approx 2d^2 \ln d$. They also obtained improved bounds for point sets on algebraic varieties of bounded degree, among other cases.

Bradford and Capoyleas [5] showed that if S is, in some sense, uniformly distributed on the $(d-1)$-dimensional sphere, then S has a weak $\frac{1}{r}$-net of size $O(r \log^2 r)$ (with the constant of proportionality depending on d).

(Aronov et al. [1] have tackled the weak ϵ-net problem from another angle, for the planar case: They seek to determine, given an integer $k \geq 1$, the maximum value r_k for which every set $S \subset \mathbb{R}^2$ has a weak $\frac{1}{r_k}$-net of size k. They derive upper and lower bounds for r_k, for small values of k. Babazadeh and Zarrabi-Zadeh [4] extended this work to the case $d = 3$.

Mustafa and Ray [13] have found a connection between weak ϵ-nets with respect to convex sets, and "strong" ϵ-nets with respect to other set systems with finite VC-dimension.)

Algorithmic aspects. The constructions of Matoušek and Wagner [12] yield an algorithm for building, for a given n-point set $S \subset \mathrm{R}^d$, $d \geq 2$, a weak $\frac{1}{r}$-net of size $O(r^d \mathrm{polylog}(r))$ in time $O(n \log r)$. For the case $d = 2$, a weak $\frac{1}{r}$-net of size $O(r^2)$ can be constructed in time $O(nr^2)$, as was shown earlier by Chazelle et al. [6].

Chazelle et al. [6] also show how to determine, in time $O(n^3)$, the largest r for which a given set N is a weak $\frac{1}{r}$-net of a given planar n-point set S. There is no known polynomial-time algorithm for this problem for dimensions 3 and larger.

Our results. In this paper we derive improved upper bounds for two of the above-mentioned cases: namely, for planar point sets in convex position, and for point sets along the moment curve μ_d (1.1). Our bounds involve the inverse Ackermann function $\alpha(r)$, and are as follows:

THEOREM 1.1. *Let S be an n-point set in convex position in the plane. Then, S has a weak $\frac{1}{r}$-net of size $O(r\alpha(r))$.*

THEOREM 1.2. *Let S be a set of n points along the d-dimensional moment curve μ_d, $d \geq 3$. Let*

$$j = \begin{cases} (d^2+d)/2, & d \text{ even;} \\ (d^2+1)/2, & d \text{ odd;} \end{cases}$$

and let $s = \lfloor (j-2)/2 \rfloor$. Then, S has a weak $\frac{1}{r}$-net of size

$$\begin{array}{ll} r \cdot 2^{O(\alpha(r)^s)}, & j \text{ even;} \\ r \cdot 2^{O(\alpha(r)^s \log \alpha(r))}, & j \text{ odd.} \end{array}$$

(Note that j is even if and only if d is divisible by 4.)

Furthermore, these weak $\frac{1}{r}$-nets can be easily constructed in time $O(n \log r)$.

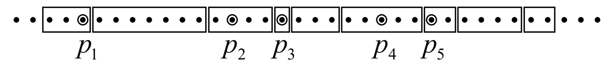

Figure 1: A 9-chain stabbed by a 5-tuple.

1.1 The inverse Ackermann function. We briefly introduce (our version of) the inverse Ackermann functions $\alpha_k(x)$ and $\alpha(x)$.

The *inverse Ackermann hierarchy* is defined as follows. Let $\alpha_1(x) = x/2$, and for each $k \geq 2$, let $\alpha_k(x)$ be the number of times we have to apply α_{k-1}, starting from x, until we reach a value not larger than 1. In other words, for $k \geq 2$, let

$$\alpha_k(x) = \begin{cases} 0, & \text{if } x \leq 1; \\ 1 + \alpha_k(\alpha_{k-1}(x)), & \text{otherwise.} \end{cases}$$

We have $\alpha_2(x) = \lceil \log_2 x \rceil$ for $x \geq 1$, and $\alpha_3(x) = \log^* x$. (Note that $\alpha_k(x)$ is always an integer for $k \geq 2$.)

Now, for every fixed $x \geq 6$, the sequence $\alpha_1(x), \alpha_2(x), \alpha_3(x), \ldots$ decreases strictly until it settles at 3. The *inverse Ackermann function*[2] $\alpha(x)$ assigns to each real number x the smallest integer k for which $\alpha_k(x) \leq 3$:

$$\alpha(x) = \min\{k \mid \alpha_k(x) \leq 3\}.$$

1.2 Interval chains. Our constructions of weak ϵ-nets follow by a reduction to a new problem, which we call *stabbing interval chains*.

Let $[i, j]$ denote the interval of integers $\{i, i+1, \ldots, j\}$; the case $i = j$ is also denoted as $[i]$. An *interval chain*[3] of size k (also called a *k-chain*) is a sequence of k consecutive, disjoint, nonempty intervals

$$C = I_1 I_2 \cdots I_k = [a_1, a_2][a_2+1, a_3] \cdots [a_k+1, a_{k+1}],$$

where $a_1 \leq a_2 < a_3 < \cdots < a_{k+1}$. We say that a j-tuple of integers $(p_1, \ldots, p_j)$ *stabs* an interval chain C if each p_i lies in a different interval of C (see Figure 1).

Our problem is to stab, with as few j-tuples as possible, all interval chains of size k that lie within a given range $[1, n]$.

DEFINITION 1.3. *Let $z_k^{(j)}(n)$ denote the minimum size of a collection $\mathcal{Z}$ of j-tuples that stab all k-chains that lie in $[1, n]$.*

[2] We follow Seidel [14, slide 85]. The function $\alpha(x)$ is usually defined slightly differently (see, for example, [11, p. 173], though there are other versions), but all variants are equivalent up to an additive constant.

[3] An identical definition of interval chains has already been given by Condon and Saks [9], for an unrelated application.

Note that $z_k^{(j)}(n)$ is increasing in n, decreasing in k, and increasing in j.

In this paper we derive almost-tight upper and lower bounds for $z_k^{(j)}(n)$, involving functions in the inverse Ackermann hierarchy. Our upper bounds for $z_k^{(j)}(n)$ are used in the proofs of Theorems 1.1 and 1.2 above. The case $j=3$ (which is the one needed for Theorem 1.1) is simpler (and tighter) than the general case $j \geq 4$, and we treat this case separately, both in the upper and the lower bounds.

Our bounds for stabbing interval chains are as follows:

THEOREM 1.4. *$z_k^{(3)}(n)$ satisfies the following bounds:*

$$z_3^{(3)}(n) = \binom{n-1}{2}; \quad z_4^{(3)}(n) = \Theta(n \log n);$$
$$z_5^{(3)}(n) = \Theta(n \log \log n);$$

and, for every $k \geq 6$, we have

$$z_k^{(3)}(n) \leq cn\alpha_{\lfloor k/2 \rfloor}(n) \quad \text{for all } n;$$
$$z_k^{(3)}(n) \geq c'n\alpha_{\lfloor k/2 \rfloor}(n) \quad \text{for all } n \geq n_k;$$

for some absolute constants c and c', and some constants n_k depending on k.

THEOREM 1.5. *Let $j \geq 4$ be fixed, and let $s = \lfloor (j-2)/2 \rfloor$. Then there exist functions $P'_j(m)$, $Q'_j(m)$, both of the form*

$$(1.2) \quad P'_j(m), Q'_j(m) = \begin{cases} 2^{(1/s!)m^s + O(m^{s-1})}, & j \text{ even;} \\ 2^{(1/s!)m^s \log_2 m + O(m^s)}, & j \text{ odd;} \end{cases}$$

such that, for every $m \geq 2$, we have

$$z_{P'_j(m)}^{(j)}(n) \leq cn\alpha_m(n) \quad \text{for all } n;$$
$$z_{Q'_j(m)}^{(j)}(n) \geq n\alpha_m(n) \quad \text{for all } n \geq n_m.$$

Here $c = c(j)$ is a constant that depends only on j, and $n_m = n_m(j)$ are constants that depend on j and m.

Thus, for every fixed j, once k is sufficiently large, $z_k^{(j)}(n)$ becomes barely superlinear in n. Moreover, if we let k grow as an appropriate function of $\alpha(n)$, then the upper bounds become *linear*. Namely, we have $z_k^{(3)}(n) = O(n)$ for $k \geq 2\alpha(n)$; and for $j \geq 4$, we have $z_k^{(j)}(n) = O(n)$ for $k \geq P'_j(\alpha(n))$.

In the full version of this paper we also prove the following bound for stabbing with *pairs* ($j = 2$):

LEMMA 1.6. *We have*

$$\frac{n}{\lfloor k/2 \rfloor} - 3 \leq z_k^{(2)}(n) \leq \frac{n}{\lfloor k/2 \rfloor} - 1.$$

2 From weak ϵ-nets to interval chains

In this section we present constructions of weak ϵ-nets that reduce to problems of stabbing interval chains with j-tuples. We first address the case when S is planar and in convex position, and then we tackle the case where S lies on the moment curve in $\mathbb{R}^d$ (as well as some related cases).

LEMMA 2.1. *Let S be a set of n points in convex position in the plane, and let $r > 1$. Then S has a weak $\frac{1}{r}$-net of size $z_{\ell/r-1}^{(3)}(\ell)$, where ℓ is a free parameter with $4r \leq \ell < n$.*

Proof. Partition the points of S into ℓ "blocks" $B_0, B_1, \ldots, B_{\ell-1}$ of n/ℓ consecutive points, clockwise along the boundary of $\mathcal{CH}(S)$. Construct a set of points $P = \{p_0, p_1, \ldots, p_{\ell-1}\}$, where each p_j lies on the boundary of $\mathcal{CH}(S)$ between the last point of B_{j-1} and the first point of B_j. (Indices are modulo ℓ. See Figure 2(a).)

Consider a subset $S' \subset S$ of size at least n/r. S' must contain $m = \ell/r$ points $q_0, q_1, \ldots, q_{m-1}$ lying on m distinct blocks. Let B_{j_k} be the block containing q_k, with $0 \leq j_0 < j_1 < \cdots < j_{m-1} < \ell$. The blocks B_{j_k} partition P cyclically into m nonempty intervals

$$I_k = \{p_{j_k+1}, p_{j_k+2}, \ldots, p_{j_{k+1}}\}, \quad \text{for } 0 \leq k < m.$$

(Indices are modulo ℓ or modulo m as appropriate.) Let $p_a, p_b, p_c, p_d \in P$ be four points belonging to four different intervals I_k, listed in cyclic order. Then the intersection between the segments $p_a p_c$ and $p_b p_d$ must lie inside $\mathcal{CH}(q_0, \ldots, q_{m-1}) \subseteq \mathcal{CH}(S')$. See Figure 2($b$).[4]

Thus, it is enough to construct a set of quadruples of points of P, such that, no matter how P is cyclically partitioned into m intervals $I_0 I_1 \cdots I_{m-1}$, some quadruple will "stab" four different intervals. The set of chord-intersection points corresponding to these quadruples is our desired weak $\frac{1}{r}$-net.

We take point p_0 as the first point for all the quadruples; by construction, p_0 lies in the last interval I_{m-1}. Thus, it only remains to build a family $\mathcal{Z}$ of *triples* of the form (p_a, p_b, p_c), with $1 \leq a < b < c < \ell$, such that some triple is guaranteed to fall on three distinct intervals among $I_0, \ldots, I_{m-2}$, in any given cyclic chain $I_0, \ldots, I_{m-1}$.

But this is isomorphic to the problem of stabbing all $(m-1)$-chains in $[1, \ell-1]$ with triples. Thus, there exists a family $\mathcal{Z}$ of size at most $z_{m-1}^{(3)}(\ell) = z_{\ell/r-1}^{(3)}(\ell)$. $\square$

Proof of Theorem 1.1. By Theorem 1.4 we have

$$(2.3) \qquad z_{\ell/r-1}^{(3)}(\ell) = O(\ell \alpha_{\ell/(2r)-1}(\ell)).$$

[4] This basic idea, initially observed by Emo Welzl, already appears in [7].

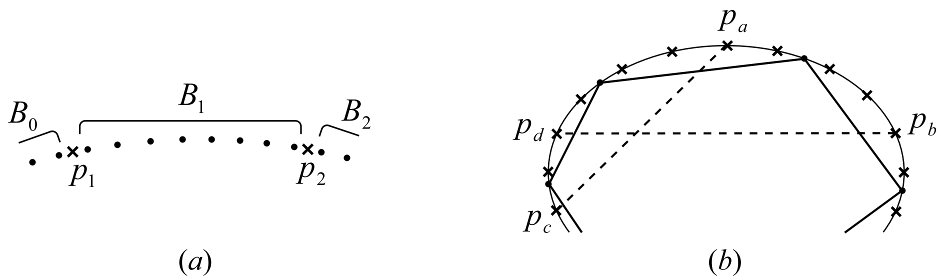

Figure 2: (a) "Separator" points p_j between consecutive blocks. (b) The intersection between two chords joining pairs of points from four different intervals falls inside $\mathcal{CH}(S')$.

We take $\ell = 2r(1+\alpha(r))$, so $\ell/(2r) - 1 = \alpha(r)$. It can be shown that $\alpha_{\alpha(r)}(\ell) \le 4$ for all large enough r (recall that $\alpha_{\alpha(r)}(r) \le 3$ by definition). Hence, (2.3) becomes $O(r\alpha(r))$. $\square$

2.1 Point sets along the moment curve. A similar reduction applies to the case when S is a set of n points along the moment curve μ_d (1.1). This curve has the property that every hyperplane intersects it in at most d points (see, e.g., Matoušek [11, p. 97]).

If A and B are two finite sets of points along μ_d, we say that A and B are *interleaving* if between every two points of A there is a point of B and vice versa. In such a case, we must have $\big||A| - |B|\big| \le 1$.

Lemma 2.2. *Let $s = \lceil (d+1)/2 \rceil$, and let $j = (s-1)(d+1)+1$. (Thus, $j = (d^2+d+2)/2$ for d even, and $j = (d^2+1)/2$ for d odd.)*

Let A be a set of j points along the moment curve $\mu_d \subset \mathbb{R}^d$. Then there exists a point $x \in \mathcal{CH}(A)$ with the following property: For every point set $B \subset \mu_d$ interleaving with A, with

$$|B| = \begin{cases} j, & d \text{ even}, \\ j+1, & d \text{ odd}, \end{cases}$$

we have $x \in \mathcal{CH}(B)$.

Proof. By Tverberg's Theorem (see, e.g., [11, p. 200]), A can be partitioned into s pairwise disjoint subsets $A_1, \ldots, A_s$, whose convex hulls all contain some common point x. This point x satisfies the assertion of the lemma, for if $x \notin \mathcal{CH}(B)$, then there would exist a hyperplane h that separates x from B. But there must be at least s points of A in the same side of h as x (at least one from each part A_i). By continuity, and since A and B are interleaving, it follows that the curve μ_d must intersect h at least $2s-1$ times if d is even, or $2s$ times if d is odd. In either case, this quantity equals $d+1$, a contradiction.[5] $\square$

The reduction from weak ϵ-nets to stabbing interval chains with j-tuples is now straightforward:

Lemma 2.3. *Let S be a set of n points along the moment curve μ_d, and let $r > 1$. Let*

$$j' = \begin{cases} (d^2+d)/2, & d \text{ even;} \\ (d^2+1)/2, & d \text{ odd.} \end{cases}$$

Then S has a weak $\frac{1}{r}$-net of size at most $z^{(j')}_{\ell/r-1}(\ell)$, where ℓ is a free parameter with $(j'+1)r \le \ell < n$.

Proof. Partition S into ℓ blocks $B_0, B_1, \ldots, B_{\ell-1}$ of n/ℓ consecutive points. Construct a set of points $P = \{p_1, \ldots, p_{\ell-1}\} \subset \mu_d$, where each p_i lies between the last point of B_{i-1} and the first point of B_i. Take also a point $p_\ell \in \mu_d$ lying after $B_{\ell-1}$.

Consider a set $S' \subset S$ of size at least n/r. S' must contain $m = \ell/r$ points $q_1, \ldots, q_m$ lying on m different blocks. These points determine an $(m-1)$-chain $C = I_1 \cdots I_{m-1}$ on P.

Thus, construct an optimal family $\mathcal{Z}'$ of j'-tuples of points in P that stab all such $(m-1)$-chains. Append the point p_ℓ to every such j'-tuple, obtaining a family $\mathcal{Z}$ of $(j'+1)$-tuples (actually, this is necessary only for d even). There must exist some $(j'+1)$-tuple $\overline{p} \in \mathcal{Z}$ whose first j' points stab the chain C. By Lemma 2.2, there exists a point $x = x(\overline{p})$ which lies in $\mathcal{CH}(q_1, \ldots, q_m) \subseteq \mathcal{CH}(S')$. Therefore, the set of all such points $x(\overline{p})$, $\overline{p} \in \mathcal{Z}$, is our desired weak $\frac{1}{r}$-net. It has size at most $z^{(j')}_{m-1}(\ell-1)$. $\square$

Proof of Theorem 1.2. Take $\ell = r(1 + P'_{j'}(\alpha(r)))$, with $P'_{j'}(m)$ as given in Theorem 1.5. Then, arguing as before,

$$z^{(j')}_{\ell/r-1}(\ell) = z^{(j')}_{P'_{j'}(\alpha(r))}(\ell) \le c\ell\alpha_{\alpha(r)}(\ell) \le 4c\ell.$$

The claim follows. $\square$

[5] The above argument is very similar to the one used by Matoušek and Wagner [12], applied to a different construction.

Remark 2.4. *These results can be generalized to curves $\gamma \subset \mathbb{R}^d$ with the property that every hyperplane intersects γ at most q times, for some integer $q \geq d$. We obtain weak $\frac{1}{r}$-nets of size $r \cdot 2^{\mathrm{poly}(\alpha(r))}$ for point sets on such curves. (The methods of [12] yield weak $\frac{1}{r}$-nets of size $O(r \operatorname{polylog}(r))$ for these point sets.)*

3 Upper bounds for stabbing interval chains

In this section we derive upper bounds on $z_k^{(j)}(n)$, the minimum number of j-tuples needed to stab all k-interval chains in $[1, n]$. We will always take j to be a constant, noting that the constants implicit in the asymptotic notations do depend on j (though neither on k nor on n).

We start with the following two simple bounds. We omit the proof of the first.

Lemma 3.1. *For all $j \geq 2$ we have*

$$z_j^{(j)}(n) = \binom{n - \lfloor j/2 \rfloor}{\lceil j/2 \rceil} = \Theta\Big(n^{\lceil j/2 \rceil}\Big).$$

Lemma 3.2. *For every fixed $j \geq 2$ we have*

$$z_{2^{j-1}}^{(j)}(n) = O\big(n \log^{j-2} n\big).$$

Proof sketch. The case $j = 2$ is given by Lemma 3.1, so let $j \geq 3$. Put $k = 2^{j-1}$. Divide the range $[1, n]$ into two blocks, each of size at most $n/2$, leaving between them the element $y = \lceil n/2 \rceil$. Recursively build, for each block, a family of j-tuples that stab all k-chains contained in the block. In addition build, for each block, a family of $(j-1)$-tuples that stab all $k/2$-chains in the block, and append the element y to each $(j-1)$-tuple.

The result is a family of j-tuples that stab all k-chains in $[1, n]$. Thus, by induction on j,

$$z_k^{(j)}(n) \leq 2 z_k^{(j)}\left(\frac{n}{2}\right) + O\big(n \log^{j-3} n\big). \qquad \square$$

We now derive upper bounds for $z_k^{(j)}(n)$ for all k. We first tackle the case $j = 3$ (the one used in the proof of Theorem 1.1), and then we briefly address the general case $j \geq 4$.

3.1 Upper bounds for triples.

We have seen that $z_3^{(3)}(n) = \binom{n-1}{2}$ (Lemma 3.1) and $z_4^{(3)}(n) = O(n \log n)$ (Lemma 3.2). Our bounds for stabbing k-chains with triples, $k \geq 5$, are based on the following recurrence relation.

Recurrence 3.3. *Let t be an integer parameter, with $1 \leq t \leq \sqrt{n/2} - 1$. Then,*

$$z_k^{(3)}(n) \leq \frac{n}{t} z_k^{(3)}(t) + z_{k-2}^{(3)}\left(\frac{n}{t}\right) + 2n.$$

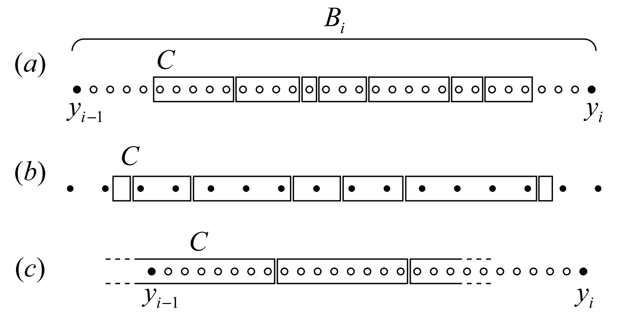

Figure 3: A k-chain C must satisfy exactly one of these three properties.

Proof. Partition the range $[1, n]$ into blocks $B_1, B_2, \ldots, B_b$ of size t (except for the last block, which might be smaller), leaving between each pair of adjacent blocks, as well as before the first block and after the last one, a single "separator" element. Let the set of separators be $Y = \{y_0, \ldots, y_b\}$, such that block B_i lies between separators y_{i-1} and y_i.

The number of blocks is $b = \left\lceil \frac{n-1}{t+1} \right\rceil$. We have $b \leq n/t - 1$, since $n \geq 2(t+1)^2 \geq 2t^2 + t$.

Now, every k-chain $C = I_1 \cdots I_k$ must satisfy exactly one of the following properties (see Figure 3):

1. C is entirely contained within a block B_i.
2. Every interval of C, except possibly the first and the last, contains a separator.
3. Some interval I_j of C, $2 \leq j \leq k-1$, falls entirely within a block B_i, and another interval of C contains either y_{i-1} or y_i.

We can take care of the first case by constructing within each block B_i an optimal family of triples that stab all k-chains. The second case is handled by constructing on the separators Y an optimal family of triples that stab all $(k-2)$-chains. And the third case is handled by taking all triples of the forms

$$\begin{aligned} (a, a+1, y_i), &\quad \text{for} \quad y_{i-1} \leq a \leq y_i - 2, \\ (y_{i-1}, a, a+1), &\quad \text{for} \quad y_{i-1} < a \leq y_i - 1, \end{aligned}$$

for all y_i. There are at most $2n$ such triples. We obtain the claimed recurrence relation. $\square$

Lemma 3.4. *We have $z_5^{(3)}(n) = O(n \log \log n)$.*

Proof. Apply Recurrence 3.3 with $k = 5$ and $t = \sqrt{n/3}$, and use Lemma 3.1. $\square$

Lemma 3.5. *There exists an absolute constant c such that, for every $k \geq 6$, we have*

$$z_k^{(3)}(n) \leq cn\alpha_{\lfloor k/2 \rfloor}(n) \quad \textit{for all } n.$$

Proof. It is convenient to work with a slight variant of the inverse Ackermann function. Let $n_0 = 2000$, and define $\widehat{\alpha}_m(x)$, $m \geq 2$, by $\widehat{\alpha}_2(x) = \alpha_2(x) = \lceil \log_2 x \rceil$, and for $m \geq 3$ by the recurrence

$$\widehat{\alpha}_m(x) = \begin{cases} 1, & \text{if } x \leq n_0; \\ 1 + \widehat{\alpha}_m(2\widehat{\alpha}_{m-1}(x)), & \text{otherwise.} \end{cases}$$

One can show (see the full version) that there exists a constant c_0 such that $|\widehat{\alpha}_m(x) - \alpha_m(x)| \leq c_0$ for all m and x.

Let $k \geq 4$, and let $m = \lfloor k/2 \rfloor$. We prove, by induction on k, that

$$z_k^{(3)}(n) \leq c_1 n \widehat{\alpha}_m(n) \quad \text{for all } n, \tag{3.4}$$

for some absolute constant c_1 (which can be assumed large enough). The base cases of the induction are $z_4^{(3)}(n), z_5^{(3)}(n) = O(n \log n)$, by Lemmas 3.2 and 3.4, respectively.

Let now $k \geq 6$, and assume (3.4) holds for $k-2$. We want to establish (3.4) for k. The case $n \leq n_0$ holds if c_1 is large enough (recall that $z_k^{(3)}(n)$ decreases with k), so assume $n > n_0$. We apply Recurrence 3.3 with $t = 2\widehat{\alpha}_{m-1}(n)$. (Note that $t \leq \sqrt{n/2} - 1$ for $n > n_0$.) Letting $z_k^{(3)}(n) = ng(n)$, observing that $\widehat{\alpha}_{m-1}(n/t) \leq \widehat{\alpha}_{m-1}(n)$, and assuming that $c_1 \geq 4$, we obtain

$$g(n) \leq g(t) + c_1.$$

Since $\widehat{\alpha}_m(t) = \widehat{\alpha}_m(n) - 1$, it follows by induction on n (with base case $n \leq n_0$) that $g(n) \leq c_1 \widehat{\alpha}_m(n)$ for all n. Therefore, (3.4) also holds for k, and we are done. $\square$

This proves the upper bounds of Theorem 1.4.

3.2 From triples to j-tuples.

We now derive upper bounds for $z_k^{(j)}(n)$, $j \geq 4$. Our bounds are based on the following recurrence relation.

Recurrence 3.6. *Let $j \geq 4$ be fixed. Let t be a parameter, $1 \leq t \leq \sqrt{n/2} - 1$, and let k_1, k_2, k_3 be integers. Put $k = 2k_1 + k_2(k_3 - 2)$. Then,*

$$\begin{aligned} z_k^{(j)}(n) \quad &\leq \quad \frac{n}{t}\left(z_k^{(j)}(t) + 2z_{k_1}^{(j-1)}(t) + z_{k_2}^{(j-2)}(t)\right) \\ &\qquad + z_{k_3}^{(j)}\left(\frac{n}{t}\right). \end{aligned}$$

Proof. Define the blocks $B_1, \ldots, B_b$ of size t and the separators $Y = \{y_0, \ldots, y_b\}$ as in the proof of Recurrence 3.3.

Let k_1, k_2, k_3 be given, and put $k = 2k_1 + k_2(k_3 - 2)$. Then, every k-chain $C = I_1 \cdots I_k$ satisfies at least one of the following properties:

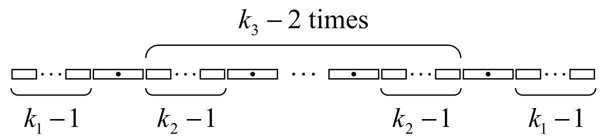

Figure 4: A chain which violates all three properties, like the one shown, can have at most $k-1$ intervals.

1. C is entirely contained within a block B_i.
2. The first k_1 intervals of C, or the last k_1 intervals of C, fall entirely within a block B_i, and some other interval of C contains the separator y_i or y_{i-1}, respectively.
3. Some k_2 consecutive intervals of C fall within a block B_i, and two other intervals contain the separators y_{i-1} and y_i.
4. At least k_3 distinct intervals of C contain separators.

Indeed, the largest number of intervals for which a chain might possibly violate *all* the above properties is

$$(k_3 - 1) + (k_3 - 2)(k_2 - 1) + 2(k_1 - 1) = k - 1.$$

(See Figure 4.) Hence, by our choice of k, one of the above properties must hold.

Thus, we can stab all k-chains by building the following family of j-tuples. Within each block B_i we build

- an optimal family of j-tuples that stab all k-chains;
- an optimal family of $(j-1)$-tuples that stab all k_1-chains, where each of these tuples is extended into a j-tuple in two ways, by appending either of the surrounding separators y_{i-1}, y_i;
- an optimal family of $(j-2)$-tuples that stab all k_2-chains, where each of these tuples is extended into a j-tuple by appending both separators y_{i-1}, y_i.

In addition, we construct on the set of separators Y an optimal family of j-tuples that stab all k_3-chains. Every k-chain C must be stabbed by some j-tuple in this family. The claimed recurrence relation follows. $\square$

Define integer-valued functions $P_j(m)$, $j, m \geq 2$, by

$$P_2(m) = 2; \quad P_3(m) = 2m;$$

and for $j \geq 4$ by

$$\begin{aligned} P_j(2) \quad &= \quad 2^{j-1}; \\ P_j(m) \quad &= \quad P_{j-2}(m)\,(P_j(m-1) - 2) \\ &\qquad + 2P_{j-1}(m), \qquad m \geq 3. \end{aligned}$$

We have $P_4(m) = 5 \cdot 2^m - 4m - 4$, and in general, letting $s = \lfloor (j-2)/2 \rfloor$, one can show that

$$P_j(m) = \begin{cases} 2^{(1/s!)m^s + O(m^{s-1})}, & \text{for } j \text{ even;} \\ 2^{(1/s!)m^s \log_2 m + O(m^s)}, & \text{for } j \text{ odd.} \end{cases}$$

LEMMA 3.7. *Let $j \geq 2$ be fixed. Then, there exists a constant $c = c(j)$ such that, for every $m \geq 2$, we have*

$$z^{(j)}_{P_j(m)}(n) \leq cn\alpha_m(n)^{j-2} \quad \textit{for all } n. \tag{3.5}$$

Proof sketch. We proceed by induction on j, and for each j by induction on m. The case $j = 3$ is given by Lemmas 3.2 and 3.5, and the case $m = 2$ is given by Lemma 3.2.

For the induction on m, we apply Recurrence 3.6 with parameters

$$k_1 = P_{j-1}(m), \quad k_2 = P_{j-2}(m), \quad k_3 = P_j(m-1),$$
$$k = P_j(m), \quad t = 4\widehat{\alpha}_{m-1}(n)^{j-2},$$

where $\widehat{\alpha}_m(x)$ is another slight variant of $\alpha_m(x)$, with recurrence $\widehat{\alpha}_m(x) = 1 + \widehat{\alpha}_m\big(4\widehat{\alpha}_{m-1}(x)^{j-2}\big)$ and base case $\widehat{\alpha}_2(x) = \alpha_2(x)$. See the full version for more details. □

Let $P'_j(m) = P_j(m+1)$ for $j \geq 4$, $m \geq 2$. Clearly, $P'_j(m)$ satisfies (1.2). There exists a constant c', depending only on j, such that $\alpha_{m+1}(n)^{j-2} \leq c'\alpha_m(n)$ for all m and n. Therefore,

$$z^{(j)}_{P'_j(m)}(n) \leq c''n\alpha_m(n) \quad \text{for all } n,$$

for some constant $c'' = c''(j)$. This proves the upper bounds of Theorem 1.5.

Computational aspects. These upper bounds for $z^{(j)}_k(n)$ yield algorithms for constructing stabbing families of j-tuples in linear time in the size of the output. Thus, the weak $\frac{1}{r}$-nets of Theorems 1.1 and 1.2 can be easily built in time $O(n \log r)$, for a given n-point set S with the appropriate properties. Consider first the planar case (of Theorem 1.1):

Let $S = (q_0, \ldots, q_{n-1})$ be a given list of n points in the plane in convex position (listed in no particular order). We can build the ℓ-point list $P = (p_0, \ldots, p_{\ell-1})$, as given in the proof of Lemma 2.1, in time $O(n \log \ell)$; we do this by divide and conquer, applying linear-time selection on each step. From the list P, we can obtain our desired weak $\frac{1}{r}$-net, of size $O(\ell) = O(r\alpha(r))$, in time $O(\ell)$. Thus, the total running time is $O(\ell + n \log \ell) = O(n \log r)$. (We may assume that $\ell \leq n$, for otherwise we can just return S itself as the desired weak $\frac{1}{r}$-net.)

The case of the moment curve is analogous. (Finding the point x of Lemma 2.2 involves examining a finite number of partitions—a constant-time operation, since d is constant.)

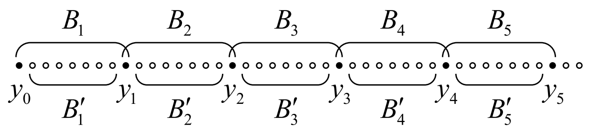

Figure 5: Blocks and contracted blocks defined in $[1, n]$.

4 Lower bounds for stabbing interval chains

We now derive asymptotic lower bounds for $z^{(j)}_k(n)$. As before, we take j to be fixed, recalling that the implicit constants do depend on j. We start with the following simple bound.

LEMMA 4.1. *For every fixed $j \geq 3$ we have*

$$z^{(j)}_{(j-1)^2}(n) = \Omega(n \log n),$$

where the constant of proportionality depends on j.

Proof. Let $t = \lceil n/j \rceil$. Define on the range $[1, n]$ a sequence $B_1, \ldots, B_j$ of j blocks of size t, where every two consecutive blocks B_i, B_{i+1} overlap at one element y_i. Also define "contracted" blocks $B'_1, \ldots, B'_j$ of size $t-2$, which do not contain the elements y_i. See Figure 5.

Let $k = (j-1)^2$, and let $\mathcal{Z}$ be a family of j-tuples that stab all k-chains in $[1, n]$. $\mathcal{Z}$ *must* contain, for each block B_i, a complete "local" family of stabbing tuples. Further, these local families are pairwise disjoint.

In addition, $\mathcal{Z}$ must contain "global" tuples—tuples not entirely contained in any B_i. Call an element $x \in B'_i$ *unused* if x is not contained in any global tuple of $\mathcal{Z}$.

Suppose each of B'_1, B'_j contains a run of $j-2$ consecutive unused elements, and each block $B'_2, \ldots, B'_{j-1}$ contains a run of $j-3$ consecutive unused elements. Construct a chain C that has these unused elements as singleton intervals, plus $j-1$ "long" intervals between the runs. Note that the long intervals are nonempty, since each contains an element y_i.

The chain C has $j^2 - 2j + 1 = k$ intervals, but it cannot be stabbed by any tuple in $\mathcal{Z}$: It cannot be stabbed by a local tuple, since each block B_i contains parts of at most $j-1$ intervals; and it cannot be stabbed by a global tuple, since the global tuples can stab only the long intervals, which number only $j-1$.

Therefore, there cannot exist such runs of unused elements. Hence, $\mathcal{Z}$ must contain $\Omega(n)$ global tuples: At the very least, there must be some B'_i in which every $(j-2)$-nd element is "used" by some global tuple.

We obtain the recurrence relation

$$z^{(j)}_k(n) \geq j z^{(j)}_k\left(\frac{n}{j}\right) + \Omega(n).$$

Thus, $z^{(j)}_k(n) = \Omega(n \log n)$. □

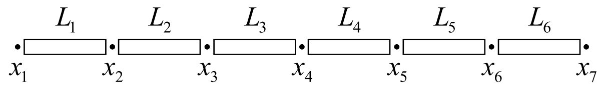

Figure 6: The m unused elements $x_1, \dots, x_m$, from m distinct blocks, define $m-1$ "links" $L_1, \dots, L_{m-1}$.

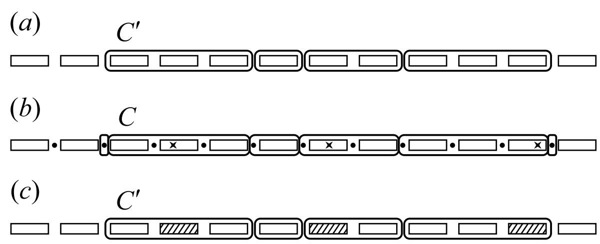

Figure 7: Every k-chain C' on the links (a) can be translated into a $(k+2)$-chain C on $[1, n]$ (b). A global triple (marked by x's) must stab C on three distinct links. We can translate this triple back into a triple of links that stabs C' (c).

4.1 Lower bounds for triples. We now derive lower bounds for $z_k^{(3)}(n)$ for all k. We use the following recurrence relation.

RECURRENCE 4.2. *Let t be an integer parameter, with $3 \le t \le \sqrt{n}$. Then,*

$$z_{k+2}^{(3)}(n) \ge \frac{n}{t} z_{k+2}^{(3)}(t) + \min\left\{\frac{n}{18},\ z_k^{(3)}\left(\frac{n}{3t}\right)\right\}$$

for all $n \ge 36$.

Proof. Let $b = \lceil n/t \rceil$. Define on $[1, n]$ a sequence $B_1, \dots, B_b$ of b blocks of size t, where every two consecutive blocks overlap at one element y_i. Also define "contracted" blocks $B'_1, \dots, B'_b$ of size $t-2$, as in the proof of Lemma 4.1. See again Figure 5.

Let $\mathcal{Z}$ be a family of triples that stab all $(k+2)$-chains in $[1, n]$. Again, $\mathcal{Z}$ must contain a complete stabbing family of "local" triples for each block B_i. $\mathcal{Z}$ must also contain "global" triples. Consider again the elements $x \in B'_i$ which are unused by the global triples.

Suppose that at most half the blocks B'_i contain unused elements. Then, the number of global triples must be at least $\frac{1}{3} \cdot \frac{b}{2}(t-2)$, which is at least $n/18$, since $t \ge 3$. In this case we are done.

Thus, suppose that at least half the blocks B'_i contain unused elements. Let $x_1, \dots, x_m$ be m unused elements from m distinct blocks, with $m \ge b/2$. These elements define a sequence of $m-1$ nonempty intervals $L_1, \dots, L_{m-1}$ between them, which we call "links" (see Figure 6).

Consider a k-chain $C' = I'_1 \cdots I'_k$ on the links, where $I'_i = [L_{a_i}, L_{a_{i+1}-1}]$ for some integers a_i, $1 \le i \le k+1$. We can translate C' into a $(k+2)$-chain $C = I_0 I_1 \cdots I_{k+1}$ on $[1, n]$, as follows: We make the unused elements right before I'_1 and after I'_k into singleton intervals, and we append each intermediate unused element to the link at its right. Then we fuse the links in each I'_i into one interval. See Figure 7(a,b).

This chain C cannot be stabbed by any local triple, since each B_i contains parts of at most two intervals of C. Thus, C must be stabbed by a global triple τ. But τ must stab three links on three different intervals among $I_1, \dots, I_k$. Thus, we can translate τ back into a triple of links τ' that stabs C'. See Figure 7(c).

Hence, $\mathcal{Z}$ must contain at least $z_k^{(3)}(m-1)$ global triples. Finally, note that $m-1 \ge n/(3t)$, since $n \ge 6\sqrt{n}$ for $n \ge 36$. The claimed recurrence relation follows. $\square$

LEMMA 4.3. *We have*

$$z_5^{(3)}(n) = \Omega(n \log\log n).$$

Proof. Apply Recurrence 4.2 with $k = 3$ and $t = \sqrt{n}$, and use Lemma 3.1. $\square$

LEMMA 4.4. *There exists an absolute constant c_1 such that, for all $k \ge 6$, we have*

$$z_k^{(3)}(n) \ge c_1 n \alpha_{\lfloor k/2 \rfloor}(n) \quad \text{for all } n \ge n_k, \tag{4.6}$$

for some integers n_k that depend on k.

Proof sketch. By induction from k to $k+2$. The base cases are $k = 6, 7$, which follow from Recurrence 4.2 by taking $k = 4$, $t = \log n$, and $k = 5$, $t = \log\log n$, respectively. Using the lower bounds of Lemmas 4.1 and 4.3, respectively, we obtain

$$z_6^{(3)}(n),\ z_7^{(3)}(n) = \Omega(n \log^* n) = \Omega(n\alpha_3(n)).$$

From here on, we use induction on k. Let $m = \lfloor k/2 \rfloor$, and assume by induction that

$$z_k^{(3)}(n) \ge c_1 n \alpha_m(n), \quad \text{for all } n \ge n_k, \tag{4.7}$$

for some constants c_1 and n_k. Assume without loss of generality that $2c_1 \le 1/18$. We apply Recurrence 4.2 with $t = \frac{1}{6}(\alpha_m(n) - 1)$. (Note that $\alpha_m(n)$ grows slowly enough that $\alpha_m(n/(3t)) \ge \alpha_m(n) - 1$ for all large enough n.) We conclude that

$$z_{k+2}^{(3)}(n) \ge c_1 n \alpha_{m+1}(n), \quad \text{for all } n \ge n_{k+2},$$

for some large enough integer n_{k+2}. $\square$

REMARK 4.5. *We cannot expect (4.6) to hold for* all n, *since $z_k^{(3)}(k) = 1$. The integers n_k provided by the proof above actually grow very fast with k.*

This proves the lower bounds of Theorem 1.4.

4.2 General lower bounds for j-tuples. We now derive general lower bounds for $z_k^{(j)}(n)$, $j \geq 4$. We will construct a sequence of integer-valued functions $Q_j(m)$, $m \geq 2$, such that

$$(4.8) \qquad z^{(j)}_{Q_j(2)}(n) = \Omega\big(n \log^{(j-1)} n\big);$$

$$(4.9) \qquad z^{(j)}_{Q_j(m)}(n) = \Omega\big(n\alpha_m^{(j-2)}(n)\big) = \omega(n\alpha_{m+1}(n)), \quad m \geq 3;$$

for all $j \geq 4$. (Here, $f^{(j)}$ denotes the j-fold composition of f.)

The case $m = 2$, given by (4.8), is based on the following recurrence relation.

Recurrence 4.6. *Let $j \geq 3$ be fixed. Let q be a parameter, with $q \leq n/(3j) - 2$. Let k_1, k_2 be integers, and put $k = 2k_1 + (j-2)k_2 + j - 1$. Then,*

$$z_k^{(j)}(n) \geq \min\left\{ \frac{n}{3jq} z_{k_1}^{(j-1)}(q),\ \frac{n}{3jq} z_{k_2}^{(j-2)}(q),\ j z_k^{(j)}\left(\frac{n}{j}\right) + \frac{n}{3j^2 q} \right\}$$

for all $n \geq 6j$.

(See the full version for the proof; it involves dividing the range $[1, n]$ into blocks of size n/j, and dividing each block into *sub-blocks* of size q.)

Now, let

$$Q_2(2) = 1; \quad Q_3(2) = 5;$$
$$Q_j(2) = 2Q_{j-1}(2) + (j-2)Q_{j-2}(2) + j - 1, \quad j \geq 4.$$

For $j \geq 4$ we have $Q_j(2) = 15, 49, 163, 577, 2139, \ldots$.

Lemma 4.7. *For every fixed $j \geq 2$ we have*

$$z^{(j)}_{Q_j(2)}(n) = \Omega\big(n \log^{(j-1)} n\big),$$

where the constant of proportionality depends on j.

Proof sketch. By induction on j. The case $j = 2$ is trivial, since $z_1^{(2)}(n) = \infty$. And the case $j = 3$ is given by Lemma 4.3. So let $j \geq 4$. Apply Recurrence 4.6 with

$$k_1 = Q_{j-1}(2), \quad k_2 = Q_{j-2}(2), \quad k = Q_j(2), \quad q = \log n.$$

Note that the recurrence relation

$$f(n) \geq j f\left(\frac{n}{j}\right) + \frac{n}{\log n}$$

has solution $f(n) = \Omega(n \log\log n)$. $\square$

The bounds (4.9) are based on the following recurrence relation.

Recurrence 4.8. *Let j be fixed. Let t and q be parameters, with $t \leq \sqrt{n}$ and $q \leq t/9 - 2$. Let k_1, k_2, k_3 be integers, and put $k = 2k_1 + (k_2+1)(k_3-1) + 1$. Then,*

$$z_k^{(j)}(n) \geq \min\left\{ \frac{n}{9q} z_{k_1}^{(j-1)}(q),\ \frac{n}{9q} z_{k_2}^{(j-2)}(q),\ \frac{n}{t} z_k^{(j)}(t) + \min\left\{ \frac{n}{9jq},\ z_{k_3}^{(j)}\left(\frac{n}{3t}\right) \right\} \right\}$$

for all $n \geq 36$.

(The proof involves dividing $[1, n]$ into blocks of size t, and dividing each block into sub-blocks of size q; see the full version.)

Define integer-valued functions $Q_j(m)$, $j, m \geq 2$, by

$$Q_2(m) = 1; \quad Q_3(m) = 2m + 1;$$

and for $j \geq 4$,

$$Q_j(m) = \big(1 + Q_{j-2}(m)\big)\big(Q_j(m-1) - 1\big) + 2Q_{j-1}(m) + 1, \quad m \geq 3;$$

with $Q_j(2)$ as defined above.

We have $Q_4(m) = 8 \cdot 2^m - 4m - 9$, and in general, letting $s = \lfloor (j-2)/2 \rfloor$, one can show that

$$Q_j(m) = \begin{cases} 2^{(1/s!)m^s + O(m^{s-1})}, & \text{for } j \geq 4 \text{ even;} \\ 2^{(1/s!)m^s \log_2 m + O(m^s)}, & \text{for } j \geq 3 \text{ odd;} \end{cases}$$

just as in the case of $P_j(m)$.

Lemma 4.9. *For every $j \geq 2$ and $m \geq 3$ we have*

$$z^{(j)}_{Q_j(m)}(n) = \Omega\big(n\alpha_m^{(j-2)}(n)\big)$$

(where the implicit constants might depend on both m and j).

Proof sketch. The case $j = 2$ is trivial, and the case $j = 3$ is given by Lemma 4.4. So let $j \geq 4$. We apply Recurrence 4.8 with the following parameters:

$$k_1 = Q_{j-1}(m), \quad k_2 = Q_{j-2}(m), \quad k_3 = Q_j(m-1), \quad k = Q_j(m).$$

We first handle the case $m = 3$, by induction on j. For this, let $t = \log^{(j-1)} n$ and $q = \alpha_3(n)$. Note that the recurrence relation

$$(4.10) \qquad f(n) \geq \frac{n}{t} f(t) + \frac{n}{q},$$

for our choice of t and q, gives $f(n) = \Omega(n \log \alpha_3(n))$, since $\alpha_3\big(\log^{(i)} n\big) = \alpha_3(n) - i$.

Then we handle the general case $m \geq 4$ by induction, setting $t = \alpha_{m-1}^{(j-2)}(n)$ and $q = \alpha_m(n)$. Now the recurrence relation (4.10) has solution $f(n) = \Omega(n \log \alpha_m(n))$. $\square$

Define $Q'_j(m)$ for $j \geq 4$, $m \geq 2$, by

$$\begin{aligned} Q'_j(2) &= j; \\ Q'_j(m) &= Q_j(m-1), \quad m \geq 3. \end{aligned}$$

Then, using the fact that $\alpha_{m-1}^{(j-1)}(n) = \omega(\alpha_m(n))$ for $m \geq 2$, we conclude by Lemmas 3.1, 4.7, and 4.9 that

$$z^{(j)}_{Q'_j(m)}(n) = \omega(n\alpha_m(n)), \quad \text{for all } j \geq 4, m \geq 2.$$

This proves the lower bounds in Theorem 1.5.

5 Discussion

The most pressing issue is to close the gap between the bounds $\Omega(r)$ and $O(r\alpha(r))$ for the size of weak $\frac{1}{r}$-nets for planar sets in convex position. A worst-case bound of $\Theta(r\alpha(r))$ would be a major achievement, since there are no known superlinear lower bounds for weak ϵ-nets for any fixed dimension d, even for arbitrary point sets.

Another open question is how tight the bounds are for the case of point sets along the moment curve μ_d. For example, does j really have to be quadratic in d in Lemma 2.2?

It would also be nice to find the exact asymptotic form of $z_k^{(j)}(n)$ for every fixed j and k.

Our divide-and-conquer approach to the problem of stabbing interval chains with triples ($j = 3$) is very similar to the approach of Alon and Schieber [3], for a problem related to offline computation of partial sums in semigroups (see also [8, 16]). In fact, both problems have the same asymptotic bounds. (However, we are not aware of any explicit reduction between the two problems.)

Our bounds for weak ϵ-nets and for stabbing interval chains also bear a remarkable similarity to the bounds on $\lambda_s(n)$, the maximum length of a Davenport–Schinzel sequence of order s on n symbols. The upper and lower bounds for $\lambda_s(n)$, for fixed $s \geq 4$, have the form $n \cdot 2^{\mathrm{poly}(\alpha(n))}$, where the degree of the polynomial in the exponent depends linearly in s (see Sharir and Agarwal [15]). We do not see any connection between the two problems, which makes this all the more surprising. Moreover, our bounds are slightly sharper than those for $\lambda_s(n)$; this makes us believe that similar improvements can be established for the bounds on $\lambda_s(n)$.

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