Weak ε-nets and interval chains

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Abstract
We construct weak ε-nets of almost linear size for certain types of point sets. Specifically, for planar point sets in convex position we construct weak 1/ε-nets of size \( O(rα(r)) \), where \( α(r) \) denotes the inverse Ackermann function. For point sets along the moment curve in \( \mathbb{R}^d \) we construct weak 1/ε-nets of size \( r \cdot 2^{poly(α(r))} \), where the degree of the polynomial in the exponent depends (quadratically) on \( d \).

Our constructions result from a reduction to a new problem, which we call stabbing interval chains with \( j \)-tuples. Given the range of integers \( N = [1, n] \), an interval chain of length \( k \) is a sequence of \( k \) consecutive, disjoint, nonempty intervals contained in \( N \). A \( j \)-tuple \( \overline{p} = (p_1, \ldots, p_j) \) is said to stab an interval chain \( C = I_1 \cdot \cdot \cdot I_k \) if each \( p_i \) falls on a different interval of \( C \). The problem is to construct a small-size family \( Z \) of \( j \)-tuples that stabs all \( k \)-interval chains in \( N \).

Let \( z_k^{(j)}(n) \) denote the minimum size of such a family \( Z \). We derive almost-tight upper and lower bounds for \( z_k^{(j)}(n) \) for every fixed \( j \); our bounds involve functions \( α_m(n) \) of the inverse Ackermann hierarchy. Specifically, we show that for \( j = 3 \) we have \( z_k^{(3)}(n) = \Theta(nα_{k/2}(n)) \) for all \( k \geq 6 \). For each \( j \geq 4 \) we construct a pair of functions \( P_j^{(j)}(m), Q_j^{(j)}(m) \), almost equal asymptotically, such that \( z_k^{(j)}(m) = O(nα_m(n)) \)

1 Introduction
Let \( S \) be an \( n \)-point set in \( \mathbb{R}^d \), and let \( ε \) be a real number, \( 0 < ε < 1 \). A weak ε-net for \( S \) (with respect to convex sets) is a set of points \( N \subset \mathbb{R}^d \), such that every convex set in \( \mathbb{R}^d \) that contains at least \( εn \) points of \( S \) contains a point of \( N \).† For convenience, we let \( r = 1/ε \), and we speak of weak 1/ε-nets, \( r > 1 \), so our bounds increase with \( r \).

Alon et al. [2] showed that, for every \( d \), for every finite \( S \subset \mathbb{R}^d \) and every \( r > 1 \) there exists a weak 1/ε-net of size at most \( fd(r) \), for some family of functions \( fd \), each depending only on \( r \).

The best known upper bound for the planar case is \( fd_2(r) = O(r^2) \), by Alon et al. [2] (see also Chazelle et al. [7]). For general \( d \geq 3 \) we have \( fd(r) = O(r^{d(\log r)^{c(d)}}) \), for some constants \( c(d) \). This was first shown by Chazelle et al. [7], and later on by Matoušek and Wagner [12] via an alternative, simpler technique.

On the other hand, there are no known lower bounds for fixed \( d \), besides the trivial \( fd(r) = Ω(r) \). (Matoušek [10] showed, though, that \( fd(r) \) increases exponentially in \( d \) for fixed \( r \); specifically, \( fd(50) = Ω(e^{\sqrt{d/2}}) \).)

If the points of \( S \) lie in certain special configurations, better bounds exist on the size of the weak ε-net. For example, Chazelle et al. [7] showed that if \( S \subset \mathbb{R}^2 \) is in convex position, then \( S \) has a weak 1/ε-net of size \( O(r(\log r)^{log_2 3}) = O(r(\log r)^{1.59}) \). Furthermore, if \( S \) is the vertex set of a regular \( n \)-gon, then \( S \) admits a weak 1/ε-net of size \( Θ(r) \).

The techniques of Matoušek and Wagner [12] also yield improved bounds for some special cases. That is, they showed that if the points of \( S \subset \mathbb{R}^d \) lie along the moment curve

\[
\mu_d = \{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\},
\]

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††The set \( N \) is called a weak ε-net because we do not necessarily have \( N \subseteq S \); otherwise, \( N \) would be a regular (or "strong") ε-net. The need to consider weak ε-nets here stems from the fact that the system of all convex sets in \( \mathbb{R}^d \) has infinite VC-dimension. For a full discussion, see Matoušek [11, Ch. 10].
then $S$ has a weak $\frac{1}{d}$-net of size $O(r(\log r)^{c(d)})$, for some constants $c(d) \approx 2d^2 \ln d$. They also obtained improved bounds for point sets on algebraic varieties of bounded degree, among other cases.

Bradford and Capoyleas [5] showed that if $S$ is, in some sense, uniformly distributed on the $(d - 1)$-dimensional sphere, then $S$ has a weak $\frac{1}{r}$-net of size $O(r(\log^2 r)$ (with the constant of proportionality depending on $d$).

(Aronov et al. [1] have tackled the weak $\epsilon$-net problem from another angle, for the planar case: They seek to determine, given an integer $k \geq 1$, the maximum value $r_k$ for which every set $S \subset \mathbb{R}^2$ has a weak $\frac{1}{r_k}$-net of size $k$. They derive upper and lower bounds for $r_k$, for small values of $k$. Babazadeh and Zarrabi-Zadeh [4] extended this work to the case $d \geq 3$.

Mustafa and Ray [13] have found a connection between weak $\epsilon$-nets with respect to convex sets, and “strong” $\epsilon$-nets with respect to other set systems with finite VC-dimension.)

**Algorithmic aspects.** The constructions of Matoušek and Wagner [12] yield an algorithm for building, for a given $n$-point set $S \subset \mathbb{R}^d$, $d \geq 2$, a weak $\frac{1}{r}$-net of size $O(r^{d+polylog}(r))$ in time $O(n \log r)$. For the case $d = 2$, a weak $\frac{1}{r}$-net of size $O(r^2)$ can be constructed in time $O(nr^2)$, as was shown earlier by Chazelle et al. [6].

Chazelle et al. [6] also show how to determine, in time $O(n^3)$, the largest $r$ for which a given set $N$ is a weak $\frac{1}{r}$-net of a given planar $n$-point set $S$. There is no known polynomial-time algorithm for this problem for dimensions 3 and larger.

**Our results.** In this paper we derive improved upper bounds for two of the above-mentioned cases: namely, for planar point sets in convex position, and for point sets along the moment curve $\mu_d$ (1.1). Our bounds involve the inverse Ackermann function $\alpha(r)$, and are as follows:

**Theorem 1.1.** Let $S$ be an $n$-point set in convex position in the plane. Then, $S$ has a weak $\frac{1}{r}$-net of size $O(r \alpha(r))$.

**Theorem 1.2.** Let $S$ be a set of $n$ points along the $d$-dimensional moment curve $\mu_d$, $d \geq 3$. Let

$$j = \begin{cases} (d^2 + d)/2, & \text{if } d \text{ even;} \\ (d^2 + 1)/2, & \text{if } d \text{ odd; } \end{cases}$$

and let $s = \lfloor (j - 2)/2 \rfloor$. Then, $S$ has a weak $\frac{1}{r}$-net of size

$$r \cdot 2^{O(\alpha(r)^{-r})}, \quad j \text{ even;}$$

$$r \cdot 2^{O(\alpha(r)^{-\log \alpha(r)})}, \quad j \text{ odd.}$$

(Note that $j$ is even if and only if $d$ is divisible by 4.)

Furthermore, these weak $\frac{1}{r}$-nets can be easily constructed in time $O(n \log r)$.

**1.1 The inverse Ackermann function.** We briefly introduce (our version of) the inverse Ackermann functions $\alpha_k(x)$ and $\alpha(x)$.

The **inverse Ackermann hierarchy** is defined as follows. Let $\alpha_1(x) = x/2$, and for each $k \geq 2$, let $\alpha_k(x)$ be the number of times we have to apply $\alpha_{k-1}$, starting from $x$, until we reach a value not larger than 1. In other words, for $k \geq 2$, let

$$\alpha_k(x) = \begin{cases} 0, & \text{if } x \leq 1; \\ 1 + \alpha_k(\alpha_{k-1}(x)), & \text{otherwise.} \end{cases}$$

We have $\alpha_2(x) = \lfloor \log_2 x \rfloor$ for $x \geq 1$, and $\alpha_3(x) = \log^* x$. (Note that $\alpha_k(x)$ is always an integer for $k \geq 2$.)

Now, for every fixed $x \geq 6$, the sequence $\alpha_1(x), \alpha_2(x), \alpha_3(x), \ldots$ decreases strictly until it settles at 3. The **inverse Ackermann function** $\alpha(x)$ assigns to each real number $x$ the smallest integer $k$ for which $\alpha_k(x) \leq 3$:

$$\alpha(x) = \min \{k \mid \alpha_k(x) \leq 3\}.$$

**1.2 Interval chains.** Our constructions of weak $\epsilon$-nets follow by a reduction to a new problem, which we call **stabbing interval chains**.

Let $[i, j]$ denote the interval of integers $\{i, i + 1, \ldots, j\}$; the case $i = j$ is also denoted as $[i]$. An interval chain$^3$ of size $k$ (also called a $k$-chain) is a sequence of $k$ consecutive, disjoint, nonempty intervals

$$C = I_1 I_2 \cdots I_k = [a_1, a_2] [a_2 + 1, a_3] \cdots [a_k + 1, a_{k+1}],$$

where $a_1 \leq a_2 < a_3 < \cdots < a_{k+1}$. We say that a $j$-tuple of integers $(p_1, \ldots, p_j)$ stabs an interval chain $C$ if each $p_j$ lies in a different interval of $C$ (see Figure 1).

![Figure 1: A 9-chain stabbed by a 5-tuple.](image)

Our problem is to stab, with as few $j$-tuples as possible, all interval chains of size $k$ that lie within a given range $[1, n]$.

**Definition 1.3.** Let $s^{(j)}(n)$ denote the minimum size of a collection $Z$ of $j$-tuples that stab all $k$-chains that lie in $[1, n]$.

$^2$We follow Seidel [14, slide 85]. The function $\alpha(x)$ is usually defined slightly differently (see, for example, [11, p. 173], though there are other versions), but all variants are equivalent up to an additive constant.

$^3$An identical definition of interval chains has already been given by Condon and Saks [9], for an unrelated application.
Note that \( z_k^{(j)}(n) \) is increasing in \( n \), decreasing in \( k \), and increasing in \( j \).

In this paper we derive almost-tight upper and lower bounds for \( z_k^{(j)}(n) \), involving functions in the inverse Ackermann hierarchy. Our upper bounds for \( z_k^{(j)}(n) \) are used in the proofs of Theorems 1.1 and 1.2 above. The case \( j = 3 \) (which is the one needed for Theorem 1.1) is simpler (and tighter) than the general case \( j \geq 4 \), and we treat this case separately, both in the upper and the lower bounds.

Our bounds for stabbing interval chains are as follows:

**Theorem 1.4.** \( z_k^{(3)}(n) \) satisfies the following bounds:

\[
\begin{aligned}
z_k^{(3)}(n) &= \begin{cases} 
\frac{n-1}{2} & j = 3, \\
\Theta(n \log n) & j = 4, \\
\Theta(n \log \log n) & j = 5, \\
\end{cases} \\
z_k^{(3)}(n) &= \Omega(n/r) \\
\end{aligned}
\]

and, for every \( k \geq 6 \), we have

\[
\begin{aligned}
z_k^{(3)}(n) &\leq cn_1(n) & \text{for all } n; \\
z_k^{(3)}(n) &\geq c'n_2(n) & \text{for } n \geq n_k; \\
\end{aligned}
\]

for some absolute constants \( c \) and \( c' \), and some constants \( n_k \) depending on \( k \).

**Theorem 1.5.** Let \( j \geq 4 \) be fixed, and let \( s = \lfloor (j-2)/2 \rfloor \). Then there exist functions \( P_j'(m), Q_j'(m) \), both of the form

\[
\begin{aligned}
P_j'(m) &= \begin{cases} 
q_1(m) + O(m^{-1}) & j \text{ even}, \\
q_1(m) \log m + O(m^{-1}) & j \text{ odd}; \\
\end{cases} \\
\end{aligned}
\]

such that, for every \( m \geq 2 \), we have

\[
\begin{aligned}
z_k^{(j)}(m) &\leq cn_{m}(n) & \text{for all } n; \\
z_k^{(j)}(m) &\geq nc_{m}(n) & \text{for } n \geq n_m. \\
\end{aligned}
\]

Here \( c = c(j) \) is a constant that depends only on \( j \), and \( n_m(n) \) are constants that depend on \( j \) and \( m \).

Thus, for every fixed \( j \), once \( k \) is sufficiently large, \( z_k^{(j)}(n) \) becomes barely superlinear in \( n \). Moreover, if we let \( k \) grow as an appropriate function of \( \alpha(n) \), then the upper bounds become linear. Namely, we have

\[
z_k^{(3)}(n) = O(n) \quad \text{for } k \geq 2\alpha(n); \quad \text{and for } j \geq 4, \quad \text{we have} \\
z_k^{(j)}(n) = O(n) \quad \text{for } k \geq P_j'(\alpha(n)). \\
\]

In the full version of this paper we also prove the following bound for stabbing with pairs \( j = 2 \):

**Lemma 1.6.** We have

\[
\frac{n}{k/2} - 3 \leq z_k^{(2)}(n) \leq \frac{n}{k/2} - 1.
\]

2 From weak \( \epsilon \)-nets to interval chains

In this section we present constructions of weak \( \epsilon \)-nets that reduce to problems of stabbing interval chains with \( j \)-tuples. We first address the case when \( S \) is planar and in convex position, and then we tackle the case where \( S \) lies on the moment curve in \( \mathbb{R}^d \) (as well as some related cases).

**Lemma 2.1.** Let \( S \) be a set of \( n \) points in convex position in the plane, and let \( r > 1 \). Then \( S \) has a weak \( \frac{1}{r} \)-net of size \( z_{\ell/r-1}^{(3)}(\ell) \), where \( \ell \) is a free parameter with \( 4r \leq \ell < n \).

**Proof.** Partition the points of \( S \) into \( \ell \) “blocks” \( B_0, B_1, \ldots, B_{\ell-1} \) of \( n/\ell \) consecutive points, clockwise along the boundary of \( CH(S) \). Construct a set of points \( P = \{p_0, p_1, \ldots, p_{\ell-1}\} \), where each \( p_j \) lies on the boundary of \( CH(S) \) between the last point of \( B_j \) and the first point of \( B_{j+1} \) (indices are modulo \( \ell \). See Figure 2(a).)

Consider a subset \( S' \subset S \) of size at least \( n/r \). \( S' \) must contain \( m = \ell/r \) points \( q_0, q_1, \ldots, q_{m-1} \) lying on \( m \) distinct blocks. Let \( B_{j_k} \) be the block containing \( q_k \), with \( 0 \leq j_0 < j_1 < \cdots < j_{m-1} < \ell \). The blocks \( B_{j_k} \) partition \( P \) cyclically into \( m \) nonempty intervals

\[
I_k = \{p_{j_k+1}, p_{j_k+2}, \ldots, p_{j_k+1}\}, \quad 0 \leq k < m.
\]

(Indices are modulo \( \ell \) or modulo \( m \) as appropriate.) Let \( p_{a_k}, p_{b_k}, p_{c_k}, p_{d_k} \in P \) be four points belonging to four different intervals \( I_k \), listed in cyclic order. Then the intersection between the segments \( p_{a_k}p_{c_k} \) and \( p_{b_k}p_{d_k} \) must lie inside \( CH(q_0, \ldots, q_{m-1}) \subset CH(S') \). See Figure 2(b).\(^4\)

Thus, it is enough to construct a set of quadruples of points of \( P \), such that, no matter how \( P \) is cyclically partitioned into \( m \) intervals \( I_0 I_1 \cdots I_{m-1} \), some quadruple will “stab” four different intervals. The set of chord-intersection points corresponding to these quadruples is our desired weak \( \frac{1}{r} \)-net.

We take point \( p_0 \) as the first point for all the quadruples; by construction, \( p_0 \) lies in the last interval \( I_{m-1} \). Thus, it only remains to build a family \( Z \) of \( (p_0, p_{a_k}, p_{b_k}, p_{c_k}) \), with \( 1 \leq a < b < c < \ell \), such that some triple is guaranteed to fall on three distinct intervals among \( I_0, \ldots, I_{m-2} \), in any given cyclic chain \( I_0, \ldots, I_{m-1} \).

But this is isomorphic to the problem of stabbing all \( (m-1) \)-chains in \([1, \ell-1]\) with triples. Thus, there exists a family \( Z \) of size at most \( z_{m-1}^{(3)}(\ell) = z_{\ell/r-1}^{(3)}(\ell) \).

\[^4\text{This basic idea, initially observed by Emo Welzl, already appears in [7].}\]
We take $\ell = 2r(1 + \alpha(r))$, so $\ell/(2r) - 1 = \alpha(r)$. It can be shown that $\alpha(\ell) \leq 4$ for all large enough $r$ (recall that $\alpha(r) \leq 3$ by definition). Hence, (2.3) becomes $O(r\alpha(r))$. \[\Box\]

2.1 Point sets along the moment curve. A similar reduction applies to the case when $S$ is a set of $n$ points along the moment curve $\mu_d(1,1)$. This curve has the property that every hyperplane intersects it in at most $d$ points (see, e.g., Matoušek [11, p. 97]).

If $A$ and $B$ are two finite sets of points along $\mu_d$, we say that $A$ and $B$ are \textit{interleaving} if between every two points of $A$ there is a point of $B$ and vice versa. In such a case, we must have $|A| - |B| \leq 1$.

LEMMA 2.2. Let $s = \lceil (d + 1)/2 \rceil$, and let $j = (s - 1)(d + 1) + 1$. (Thus, $j = (d^2 + d + 2)/2$ for $d$ even, and $j = (d^2 + 1)/2$ for $d$ odd.)

Let $A$ be a set of $j$ points along the moment curve $\mu_d \subset \mathbb{R}^d$. Then there exists a point $x \in \mathcal{CH}(A)$ with the following property: For every point set $B \subset \mu_d$ interleaving with $A$, with

$$|B| = \begin{cases} j, & d \text{ even}, \\ j + 1, & d \text{ odd}, \end{cases}$$

we have $x \in \mathcal{CH}(B)$. \[\Box\]

Proof. By Tverberg’s Theorem (see, e.g., [11, p. 200]), $A$ can be partitioned into $s$ pairwise disjoint subsets $A_1, \ldots, A_s$, whose convex hulls all contain some common point $x$. This point $x$ satisfies the assertion of the lemma, for if $x \notin \mathcal{CH}(B)$, then there would exist a hyperplane $h$ that separates $x$ from $B$. But there must be at least $s$ points of $A$ in the same side of $h$ as $x$ (at least one from each part $A_i$). By continuity, and since $A$ and $B$ are interleaving, it follows that the curve $\mu_d$ must intersect $h$ at least $2s - 1$ times if $d$ is even, or $2s$ times if $d$ is odd. In either case, this quantity equals $d + 1$, a contradiction.\footnote{The above argument is very similar to the one used by Matoušek and Wagner [12], applied to a different construction.}

The reduction from weak $\epsilon$-nets to stabbing interval chains with $j'$-tuples is now straightforward:

LEMMA 2.3. Let $S$ be a set of $n$ points along the moment curve $\mu_d$, and let $r > 1$. Let

$$j' = \begin{cases} (d^2 + d)/2, & d \text{ even}; \\ (d^2 + 1)/2, & d \text{ odd}. \end{cases}$$

Then $S$ has a weak $\frac{1}{r}$-net of size at most $z_{\ell/r-1}^{(j')}$, where $\ell$ is a free parameter with $(j' + 1)r \leq \ell < n$.

\[\Box\]

Proof. Partition $S$ into $\ell$ blocks $B_0, B_1, \ldots, B_{\ell-1}$ of $n/\ell$ consecutive points. Construct a set of points $P = \{p_1, \ldots, p_{\ell-1}\} \subset \mu_d$, where each $p_i$ lies between the last point of $B_{i-1}$ and the first point of $B_i$. Take also a point $p_\ell \in \mu_d$ lying after $B_{\ell-1}$.

Consider a set $S' \subset S$ of size at least $n/r$. $S'$ must contain $m = \ell/r$ points $q_1, \ldots, q_m$ lying on $m$ different blocks. These points determine an $(m-1)$-chain $C = I_1 \cdots I_{m-1}$ on $P$.

Thus, construct an optimal family $Z'$ of $j'$-tuples of points in $P$ that stab all such $(m-1)$-chains. Append the point $p_\ell$ to every such $j'$-tuple, obtaining a family $Z$ of $(j' + 1)$-tuples (actually, this is necessary only for $d$ even). There must exist some $(j' + 1)$-tuple $\mathbf{p} \in Z$ whose first $j'$ points stab the chain $C$. By Lemma 2.2, there exists a point $x = x(\mathbf{p})$ which lies in $\mathcal{CH}(q_1, \ldots, q_m) \subseteq \mathcal{CH}(S')$. Therefore, the set of all such points $x(\mathbf{p})$, $\mathbf{p} \in Z$, is our desired weak $\frac{1}{r}$-net. It has size at most $z_{\ell-1}^{(j')}$. \[\Box\]

Proof of Theorem 1.2. Take $\ell = r(1 + P'_{j'}(\alpha(r)))$, with $P'_j(m)$ as given in Theorem 1.5. Then, arguing as before,

$$z_{\ell/r-1}^{(j')} = z_{P'_{j'}(\alpha(r))}^{(j')} \leq c\ell\alpha(\ell) \leq 4c\ell.$$

The claim follows. \[\Box\]
Remark 2.4. These results can be generalized to curves \( \gamma \subset \mathbb{R}^d \) with the property that every hyperplane intersects \( \gamma \) at most \( q \) times, for some integer \( q \geq d \). We obtain weak \( \frac{1}{r} \)-nets of size \( r \cdot 2^{\mathcal{O}(\alpha(r))} \) for point sets on such curves. (The methods of [12] yield weak \( \frac{1}{r} \)-nets of size \( O(r \cdot \text{polylog}(r)) \) for these point sets.)

3 Upper bounds for stabbing interval chains

In this section we derive upper bounds on \( z_k^{(j)}(n) \), the minimum number of \( j \)-tuples needed to stab all \( k \)-interval chains in \([1, n]\). We will always take \( j \) to be a constant, noting that the constants implicit in the asymptotic notations do depend on \( j \) (though neither on \( k \) nor on \( n \)).

We start with the following two simple bounds. We omit the proof of the first.

**Lemma 3.1.** For all \( j \geq 2 \) we have

\[
z^{(j)}_j(n) = \left( \frac{n - \lfloor j/2 \rfloor}{\lfloor j/2 \rfloor} \right) = \Theta \left( \frac{n^{j/2}}{j} \right).
\]

**Lemma 3.2.** For every fixed \( j \geq 2 \) we have

\[
z^{(j)}_{2^{j-1}}(n) = O(n \log^{j-2} n).
\]

Proof sketch. The case \( j = 2 \) is given by Lemma 3.1, so let \( j \geq 3 \). Put \( k = 2^{j-1} \). Divide the range \([1, n]\) into two blocks, each of size at most \( n/2 \), leaving between them the element \( y = \lfloor n/2 \rfloor \). Recursively build, for each block, a family of \( j \)-tuples that stab all \( k \)-chains contained in the block. In addition build, for each block, a family of \((j-1)\)-tuples that stab all \( k/2 \)-chains in the block, and append the element \( y \) to each \((j-1)\)-tuple.

The result is a family of \( j \)-tuples that stab all \( k \)-chains in \([1, n]\). Thus, by induction on \( j \),

\[
z^{(j)}_k(n) \leq 2z^{(j)}_k \left( \frac{n}{2} \right) + O(n \log^{j-3} n). \tag*{\Box}
\]

We now derive upper bounds for \( z^{(j)}_k(n) \) for all \( k \). We first tackle the case \( j = 3 \) (the one used in the proof of Theorem 1.1), and then we briefly address the general case \( j \geq 4 \).

### 3.1 Upper bounds for triples

We have seen that \( z^{(3)}_3(n) = \left( \frac{n-1}{2} \right) \) (Lemma 3.1) and \( z^{(3)}_4(n) = O(n \log n) \) (Lemma 3.2). Our bounds for stabbing \( k \)-chains with triples, \( k \geq 5 \), are based on the following recurrence relation.

**Recurrence 3.3.** Let \( t \) be an integer parameter, with \( 1 \leq t \leq \sqrt{n/2} - 1 \). Then,

\[
z^{(3)}_k(n) \leq \frac{n}{t} z^{(3)}_k(t) + z^{(3)}_{k-2} \left( \frac{n}{t} \right) + 2n.
\]

**Proof.** Partition the range \([1, n]\) into blocks \( B_1, B_2, \ldots, B_k \) of size \( t \) (except for the last block, which might be smaller), leaving between each pair of adjacent blocks, as well as before the first block and after the last one, a single “separator” element. Let the set of separators be \( Y = \{y_i, \ldots, y_t\} \), such that block \( B_i \) lies between separators \( y_{i-1} \) and \( y_i \).

The number of blocks is \( b = \left\lceil \frac{n}{t+1} \right\rceil \). We have \( b \leq n/t - 1 \), since \( n \geq 2(t+1)^2 \geq 2t^2 + t \).

Now, every \( k \)-chain \( C = I_1 \cdots I_k \) must satisfy exactly one of the following properties (see Figure 3):

1. \( C \) is entirely contained within a block \( B_i \).
2. Every interval of \( C \), except possibly the first and the last, contains a separator.
3. Some interval \( I_j \) of \( C \), \( 2 \leq j \leq k - 1 \), falls entirely within a block \( B_i \), and another interval of \( C \) contains either \( y_{i-1} \) or \( y_i \).

We can take care of the first case by constructing within each block \( B_i \) an optimal family of triples that stab all \( k \)-chains. The second case is handled by constructing on the separators \( Y \) an optimal family of triples that stab all \((k-2)\)-chains. And the third case is handled by taking all triples of the forms

\[
(a, a+1, y_i), \quad \text{for} \quad y_{i-1} \leq a \leq y_i - 2,
\]

\[
(y_{i-1}, a, a+1), \quad \text{for} \quad y_{i-1} < a \leq y_i - 1,
\]

for all \( y_i \). There are at most \( 2n \) such triples. We obtain the claimed recurrence relation. \tag*{\Box}

**Lemma 3.4.** We have \( z^{(3)}_5(n) = O(n \log \log n) \).

**Proof.** Apply Recurrence 3.3 with \( k = 5 \) and \( t = \sqrt{n/3} \), and use Lemma 3.1. \tag*{\Box}

**Lemma 3.5.** There exists an absolute constant \( c \) such that, for every \( k \geq 6 \), we have

\[
z^{(3)}_k(n) \leq c n a_k k^{2} \left( \frac{n}{k} \right) \quad \text{for all} \ n.
\]
Proof. It is convenient to work with a slight variant of the inverse Ackermann function. Let \( n_0 = 2000 \), and define \( \tilde{\alpha}_m(x) \), \( m \geq 2 \), by \( \tilde{\alpha}_2(x) = \alpha_2(x) = \lceil \log_2 x \rceil \), and for \( m \geq 3 \) by the recurrence

\[
\tilde{\alpha}_m(x) = \begin{cases} 
1, & \text{if } x \leq n_0; \\
1 + \tilde{\alpha}_m(2\tilde{\alpha}_{m-1}(x)), & \text{otherwise}. 
\end{cases}
\]

One can show (see the full version) that there exists a constant \( c_0 \) such that \( |\tilde{\alpha}_m(x) - \alpha_m(x)| \leq c_0 \) for all \( m \) and \( x \).

Let \( k \geq 4 \), and let \( m = \lfloor k/2 \rfloor \). We prove, by induction on \( k \), that

\[
(3.4) \quad z_k^{(3)}(n) \leq c_1 n \tilde{\alpha}_m(n) \quad \text{for all } n,
\]

for some absolute constant \( c_1 \) (which can be assumed large enough). The base cases of the induction are \( z_4^{(3)}(n), z_5^{(3)}(n) = O(n \log n) \), by Lemmas 3.2 and 3.4, respectively.

Let now \( k \geq 6 \), and assume (3.4) holds for \( k - 2 \). We want to establish (3.4) for \( k \). The case \( n \leq n_0 \) holds if \( c_1 \) is large enough (recall that \( z_k^{(3)}(n) \) decreases with \( k \)), so assume \( n > n_0 \). We apply Recurrence 3.3 with \( t = 2\tilde{\alpha}_{m-1}(n) \). (Note that \( t \leq \sqrt{n/2} - 1 \) for \( n > n_0 \).) Letting \( z_k^{(3)}(n) = ng(n) \), observing that \( \tilde{\alpha}_{m-1}(n/t) \leq \tilde{\alpha}_{m-1}(n) \), and assuming that \( c_1 \geq 4 \), we obtain

\[
g(n) \leq g(t) + c_1.
\]

Since \( \tilde{\alpha}_m(t) = \tilde{\alpha}_m(n) - 1 \), it follows by induction on \( n \) (with base case \( n \leq n_0 \)) that \( g(n) \leq c_1 \tilde{\alpha}_m(n) \) for all \( n \). Therefore, (3.4) also holds for \( k \), and we are done. \( \Box \)

This proves the upper bounds of Theorem 1.4.

3.2 From triples to \( j \)-tuples. We now derive upper bounds for \( z_k^{(j)}(n) \), \( j \geq 4 \). Our bounds are based on the following recurrence relation.

Recurrence 3.6. Let \( j \geq 4 \) be fixed. Let \( t \) be a parameter, \( 1 \leq t \leq \sqrt{n/2} - 1 \), and let \( k_1, k_2, k_3 \) be integers. Put \( k = 2k_1 + k_2(k_3 - 2) \). Then,

\[
z_k^{(j)}(n) \leq \frac{n}{t} \left( z_k^{(j)}(t) + 2z_k^{(j-1)}(t) + z_k^{(j-2)}(t) \right) + z_k^{(j)} \left( \frac{n}{t} \right).
\]

Proof. Define the blocks \( B_1, \ldots, B_6 \) of size \( t \) and the separators \( Y = \{y_0, \ldots, y_b\} \) as in the proof of Recurrence 3.3.

Let \( k_1, k_2, k_3 \) be given, and put \( k = 2k_1 + k_2(k_3 - 2) \). Then, every \( k \)-chain \( C = I_1 \cdots I_k \) satisfies at least one of the following properties:

1. \( C \) is entirely contained within a block \( B_i \).
2. The first \( k_1 \) intervals of \( C \), or the last \( k_1 \) intervals of \( C \), fall entirely within a block \( B_i \), and some other interval of \( C \) contains the separator \( y_j \), or \( y_{j-1} \), respectively.
3. Some \( k_2 \) consecutive intervals of \( C \) fall within a block \( B_i \), and two other intervals contain the separators \( y_{j-1} \) and \( y_j \).
4. At least \( k_3 \) distinct intervals of \( C \) contain separators.

Indeed, the largest number of intervals for which a chain might possibly violate all the above properties is

\[(k_3 - 1) + (k_3 - 2)(k_2 - 1) + 2(k_1 - 1) = k - 1.
\]

(See Figure 4.) Hence, by our choice of \( k \), one of the above properties must hold.

Thus, we can stab all \( k \)-chains by building the following family of \( j \)-tuples. Within each block \( B_i \) we build

- an optimal family of \( j \)-tuples that stab all \( k \)-chains;
- an optimal family of \((j - 1)\)-tuples that stab all \( k_1 \)-chains, where each of these tuples is extended into a \( j \)-tuple in two ways, by appending either of the surrounding separators \( y_{j-1}, y_j \);
- an optimal family of \((j - 2)\)-tuples that stab all \( k_2 \)-chains, where each of these tuples is extended into a \( j \)-tuple by appending both separators \( y_{j-1}, y_j \).

In addition, we construct on the set of separators \( Y \) an optimal family of \( j \)-tuples that stab all \( k_3 \)-chains. Every \( k \)-chain \( C \) must be stabbed by some \( j \)-tuple in this family. The claimed recurrence relation follows. \( \Box \)

Define integer-valued functions \( P_j(m), j, m \geq 2 \), by

\[
P_2(m) = 2; \quad P_3(m) = 2m;
\]

and for \( j \geq 4 \) by

\[
P_j(2) = 2^{j-1};
\]

\[
P_j(m) = P_{j-2}(m)(P_{j-2}(m) - 2) + 2P_{j-1}(m), \quad m \geq 3.
\]
We have \( P_j(m) = 5 \cdot 2^m - 4m - 4 \), and in general, letting \( s = ([j-2]/2) \), one can show that
\[
P_j(m) = \begin{cases} 
q((1/s)x^2) + O(m^{-1}), & \text{for } j \text{ even;} \\
q((1/s)x^2) + m + O(m^2), & \text{for } j \text{ odd.}
\end{cases}
\]

**Lemma 3.7.** Let \( j \geq 2 \) be fixed. Then, there exists a constant \( c = c(j) \) such that, for every \( m \geq 2 \), we have
\[
z_j^{(j)}(m)(n) \leq c\alpha_m(n)^{j-2} \quad \text{for all } n.
\]

**Proof sketch.** We proceed by induction on \( j \), and for each \( j \) by induction on \( m \). The case \( j = 3 \) is given by Lemmas 3.2 and 3.5, and the case \( m = 2 \) is given by Lemma 3.2.

For the induction on \( m \), we apply Recurrence 3.6 with parameters
\[
k_1 = P_{j-1}(m), \quad k_2 = P_{j-2}(m), \quad k_3 = P_j(m-1),
\]
\[
k = P_j(m), \quad t = 4\alpha_{m-1}(n)^{j-2},
\]
where \( \alpha_m(x) \) is another slight variant of \( \alpha_m(x) \), with recurrence \( \alpha_m(x) = 1 + \alpha_m(4\alpha_m(x)^{j-2}) \) and base case \( \alpha_2(x) = \alpha_2(2x) \). See the full version for more details. \( \Box \)

Let \( P_j^*(m) = P_j(m+1) \) for \( j \geq 4, \ m \geq 2 \). Clearly, \( P_j^*(m) \) satisfies (1.2). There exists a constant \( c' \), depending only on \( j \), such that \( \alpha_{m+1}(n)^{j-2} \leq c'\alpha_m(n) \) for all \( m \) and \( n \). Therefore,
\[
z_j^{(j)}(m)(n) \leq c'' \alpha_{m}(n) \quad \text{for all } n,
\]
for some constant \( c'' = c''(j) \). This proves the upper bounds of Theorem 1.5.

**Computational aspects.** These upper bounds for \( z_j^{(j)}(n) \) yield algorithms for constructing stabbing families of \( j \)-tuples in linear time in the size of the output. Thus, the weak \( \frac{1}{2} \)-nets of Theorems 1.1 and 1.2 can be easily built in time \( O(n \log r) \), for a given \( n \)-point set \( S \) with the appropriate properties. Consider first the planar case (of Theorem 1.1):

Let \( S = (q_0, \ldots, q_{n-1}) \) be a given list of \( n \) points in the plane in convex position (listed in no particular order). We can build the \( \varepsilon \)-point list \( P = (p_0, \ldots, p_{\ell-1}) \), as given in the proof of Lemma 2.1, in time \( O(n \log \ell) \); we do this by divide and conquer, applying linear-time selection on each step. From the list \( P \), we can obtain our desired weak \( \frac{1}{2} \)-net, of size \( O(\ell) = O(r \alpha(r)) \), in time \( O(\ell) \). Thus, the total running time is \( O(\ell + n \log \ell) = O(n \log r) \). (We may assume that \( \ell \leq n \), for otherwise we can just return \( S \) itself as the desired weak \( \frac{1}{2} \)-net.)

The case of the moment curve is analogous. (Finding the point \( x \) of Lemma 2.2 involves examining a finite number of partitions—a constant-time operation, since \( d \) is constant.)

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**Figure 5:** Blocks and contracted blocks defined in \([1, n]\).

**4 Lower bounds for stabbing interval chains**

We now derive asymptotic lower bounds for \( z_j^{(j)}(n) \). As before, we take \( j \) to be fixed, recalling that the implicit constants do depend on \( j \). We start with the following simple bound.

**Lemma 4.1.** For every fixed \( j \geq 3 \) we have
\[
z_j^{(j)}(n)(n) = \Omega(n \log n),
\]
where the constant of proportionality depends on \( j \).

**Proof.** Let \( t = \lceil n/j \rceil \). Define on the range \([1, n]\) a sequence \( B_1, \ldots, B_t \) of \( j \) blocks of size \( \ell \), where every two consecutive blocks \( B_i, B_{i+1} \) overlap at one element \( y_i \). Also define “contracted” blocks \( B'_1, \ldots, B'_t \) of size \( \ell - 2 \), which do not contain the elements \( y_i \). See Figure 5.

Let \( k = (j - 1)^2 \), and let \( Z \) be a family of \( j \)-tuples that stab all \( k \)-chains in \([1, n]\). \( Z \) must contain, for each block \( B_i \), a complete “local” family of stabbing tuples. Further, these local families are pairwise disjoint.

In addition, \( Z \) must contain “global” tuples—tuples not entirely contained in any \( B_i \). Call an element \( x \in B'_i \) **unused** if \( x \) is not contained in any global tuple of \( Z \).

Suppose each of \( B'_1, B'_2 \) contains a run of \( j - 2 \) consecutive unused elements, and each block \( B'_2, \ldots, B'_{t-1} \) contains a run of \( j - 3 \) consecutive unused elements. Construct a chain \( C \) that has these unused elements as singleton intervals, plus \( j - 1 \) “long” intervals between the runs. Note that the long intervals are nonempty, since each contains an element \( y_i \).

The chain \( C \) has \( j^2 - 2j + 1 = k \) intervals, but it cannot be stabbed by any tuple in \( Z \). It cannot be stabbed by a local tuple, since each block \( B_i \) contains parts of at most \( j - 1 \) intervals; and it cannot be stabbed by a global tuple, since the global tuples can stab only the long intervals, which number only \( j - 1 \).

Therefore, there cannot exist such runs of unused elements. Hence, \( Z \) must contain \( \Omega(n) \) global tuples: At the very least, there must be some \( B'_i \) in which every \( (j - 2) \)-nd element is “used” by some global tuple.

We obtain the recurrence relation
\[
z_k^{(j)}(n)(n) \geq jz_k^{(j)} \left( \frac{n}{j} \right) + \Omega(n).
\]

Thus, \( z_k^{(j)}(n)(n) = \Omega(n \log n) \).
\( \Box \)
We can translate this triple back into a triple of elements from \( [1, n] \). Let \( z \), \( n \), \( k \) contain unused elements. Let \( a, b, c \in \{1, \ldots, n\} \) be integers provided by the proof above actually grow very fast with \( k \).

This proves the lower bounds of Theorem 1.4.

**Remark 4.5.** We cannot expect (4.6) to hold for all \( n \), since \( z_k^{(3)}(k) = 1 \). The integers \( n \) provided by the proof above actually grow very fast with \( k \).

**Lemma 4.3.** We have
\[
z_6^{(3)}(n) = \Omega(n \log \log n).
\]

**Proof.** Apply Recurrence 4.2 with \( k = 3 \) and \( t = \sqrt{n} \), and use Lemma 3.1. □

**Lemma 4.4.** There exists an absolute constant \( c_1 \) such that, for all \( k \geq 6 \), we have
\[
z_k^{(3)}(n) \geq c_1 n \alpha(n/3t) \quad \text{for all } n \geq n_k,
\]
for some integers \( n_k \) that depend on \( k \).

**Proof sketch.** By induction from \( k \) to \( k + 2 \). The base cases are \( k = 6, 7 \), which follow from Recurrence 4.2 by taking \( k = 4 \), \( t = \log n \), and \( k = 5 \), \( t = \log \log n \), respectively. Using the lower bounds of Lemmas 4.1 and 4.3, respectively, we obtain
\[
z_6^{(3)}(n), z_7^{(3)}(n) = \Omega(n \log^* n) = \Omega(n \alpha_3(n)).
\]

From here on, we use induction on \( k \). Let \( m = \lfloor k/2 \rfloor \), and assume by induction that
\[
z_k^{(3)}(n) \geq c_1 n \alpha_m(n) \quad \text{for all } n \geq n_k,
\]
for some constants \( c_1 \) and \( n_k \). Assume without loss of generality that \( 2c_1 \leq 1/18 \). We apply Recurrence 4.2 with \( t = \frac{1}{6} (\alpha_m(n) - 1) \). (Note that \( \alpha_m(n) \) grows slowly enough that \( \alpha_m(n/3t) \geq \alpha_m(n) - 1 \) for all large enough \( n \).) We conclude that
\[
z_k^{(3)}(n) \geq c_1 n \alpha_m(n+1) \quad \text{for all } n \geq n_{k+2},
\]
for some large enough integer \( n_{k+2} \). □

**4.1 Lower bounds for triples.** We now derive lower bounds for \( z_k^{(3)}(n) \) for all \( k \). We use the following recurrence relation.

**Recurrence 4.2.** Let \( t \) be an integer parameter, with \( 3 \leq t \leq \sqrt{n} \). Then,
\[
z_k^{(3)}(n) \geq \frac{n}{t} z_{k+2}^{(3)}(t) + \min \left\{ \frac{n}{18}, z_k^{(3)} \left( \frac{n}{3t} \right) \right\}
\]
for all \( n \geq 36 \).

**Proof.** Let \( b = \lfloor n/t \rfloor \). Define on \( [1, n] \) a sequence \( B_1, \ldots, B_b \) of blocks of size \( t \), where every two consecutive blocks overlap at one element \( y_i \). Also define “contracted” blocks \( B'_1, \ldots, B'_t \) of size \( t - 2 \), as in the proof of Lemma 4.1. See again Figure 5.

Let \( Z \) be a family of triples that stab all \( (k+2) \)-chains in \( [1, n] \). Again, \( Z \) must contain a complete stabbing family of “local” triples for each block \( B_i \). \( Z \) must also contain “global” triples. Consider again the elements \( x \in B'_t \) which are unused by the global triples.

Suppose that at most half the blocks \( B'_t \) contain unused elements. Then, the number of global triples must be at least \( \frac{1}{3} \cdot \frac{2}{3} (t - 2) \), which is at least \( n/18 \), since \( t \geq 3 \). In this case we are done.

Thus, suppose that at least half the blocks \( B'_t \) contain unused elements. Let \( x_1, \ldots, x_m \) be \( m \) unused elements from \( m \) distinct blocks, with \( m \geq b/2 \). These elements define a sequence of \( m - 1 \) nonempty intervals \( L_1, \ldots, L_{m-1} \) between them, which we call “links” (see Figure 6).

Consider a \( k \)-chain \( C' = I'_1 \cdots I'_{k+1} \) on the links, where \( I'_i = [L_{a_i}, L_{a_{i+1}}] \) for some integers \( a_i, 1 \leq i \leq k + 1 \). We can translate \( C' \) into a \( (k+2) \)-chain \( C = I_0 I_1 \cdots I_{k+1} \) on \( [1, n] \), as follows: We make the unused elements right before \( I'_1 \) and after \( I'_{k+1} \) into singleton intervals, and we append each intermediate unused element to the link at its right. Then we fuse the links in each \( I'_i \) into one interval. See Figure 7(a, b).

This chain \( C \) cannot be stabbed by any local triple, since each \( B_i \) contains parts of at most two intervals of \( C \). Thus, \( C \) must be stabbed by a global triple \( \tau \). But \( \tau \) must stab three links on three different intervals among \( I_1, \ldots, I_k \). Thus, we can translate \( \tau \) back into a triple of links \( \tau' \) that stabs \( C' \). See Figure 7(c).

Hence, \( Z \) must contain at least \( z_k^{(3)}(m - 1) \) global triples. Finally, note that \( m - 1 \geq n/(3t) \), since \( n \geq 6 \sqrt{n} \) for \( n \geq 36 \). The claimed recurrence relation follows. □

**Figure 6:** The \( m \) unused elements \( x_1, \ldots, x_m \), from \( m \) distinct blocks, define \( m - 1 \) “links” \( L_1, \ldots, L_{m-1} \).

**Figure 7:** Every \( k \)-chain \( C' \) on the links \( (a) \) can be translated into a \( (k+2) \)-chain \( C \) on \( [1, n] \) \( (b) \). A global triple (marked by ‘s) must stab \( C \) on three distinct links. We can translate this triple back into a triple of links that stabs \( C' \) \( (c) \).
4.2 General lower bounds for $j$-tuples. We now derive general lower bounds for $z^{(j)}_k(n)$, $j \geq 4$. We will construct a sequence of integer-valued functions $Q_j(m)$, $m \geq 2$, such that

\begin{align*}
q^{(j)}_{Q_j(2)}(n) & = \Omega(n \log^{(j-1)} n); \\
q^{(j)}_{Q_j(m)}(n) & = \Omega(n \alpha_{m+1}^{(j-2)}(n)) \\
& = \omega(n \alpha_{m+1}^{(j-2)}(n)), \quad m \geq 3;
\end{align*}

for all $j \geq 4$. (Here, $f^{(j)}$ denotes the $j$-fold composition of $f$.)

The case $m = 2$, given by (4.8), is based on the following recurrence relation.

**Recurrence 4.6.** Let $j \geq 3$ be fixed. Let $q$ be a parameter, with $q \leq n/(3j) - 2$. Let $k_1$, $k_2$ be integers, and put $k = 2k_1 + (j - 2)k_2 + j - 1$. Then,

\[
q^{(j)}_k(n) \geq \min \left\{ \frac{n}{3jq} q^{(j-1)}(q), \frac{n}{3jq} q^{(j-2)}(q), \frac{n}{3jq} q^{(j)}(q) + \frac{n}{3jq^2} \right\}
\]

for all $n \geq 6j$.

(See the full version for the proof; it involves dividing the range $[1, n]$ into blocks of size $n/j$, and dividing each block into sub-blocks of size $q$.)

Now, let

\[
Q_2(2) = 1; \quad Q_3(2) = 5; \\
Q_j(2) = 2Q_{j-1}(2) + (j - 2)Q_{j-2}(2) + j - 1, \quad j \geq 4.
\]

For $j \geq 4$ we have $Q_j(2) = 15, 49, 163, 577, 2139, \ldots$

**Lemma 4.7.** For every fixed $j \geq 2$ we have

\[
q^{(j)}_{Q_j(2)}(n) = \Omega(n \log^{(j-1)} n),
\]

where the constant of proportionality depends on $j$.

**Proof sketch.** By induction on $j$. The case $j = 2$ is trivial, since $z^{(2)}_2(n) = \infty$. The case $j = 3$ is given by Lemma 4.3. So let $j \geq 4$. Apply Recurrence 4.6 with

\[
k_1 = Q_{j-1}(2), \quad k_2 = Q_{j-2}(2), \quad k = Q_j(2), \quad q = \log n.
\]

Note that the recurrence relation

\[
f(n) \geq jf\left(\frac{n}{j}\right) + \frac{n}{\log n}
\]

has solution $f(n) = \Omega(n \log \log n)$.

The bounds (4.9) are based on the following recurrence relation.

**Recurrence 4.8.** Let $j$ be fixed. Let $t$ and $q$ be parameters, with $t \leq \sqrt{n}$ and $q \leq t/9 - 2$. Let $k_1$, $k_2$, $k_3$ be integers, and put $k = 2k_1 + (k_2 + 1)(k_3 - 1) + 1$. Then,

\[
z^{(j)}_k(n) \geq \min \left\{ \frac{n}{9q} z^{(j-1)}_{k_1}(q), \frac{n}{9q} z^{(j-2)}_{k_2}(q), \frac{n}{9q} z^{(j)}_{k_3}(q) \right\} + \frac{n}{t} z^{(j)}_k(t)
\]

for all $n \geq 36$.

(The proof involves dividing $[1, n]$ into blocks of size $t$, and dividing each block into sub-blocks of size $q$; see the full version.)

Define integer-valued functions $Q_j(m)$, $j, m \geq 2$, by

\[
Q_2(m) = 1; \quad Q_3(m) = 2m + 1;
\]

and for $j \geq 4$,

\[
Q_j(m) = (1 + Q_{j-2}(m))(Q_j(m - 1) + 2Q_{j-1}(m) + 1), \quad m \geq 3;
\]

with $Q_j(2)$ as defined above.

We have $Q_4(m) = 8 \cdot 2^m - 4m - 9$, and in general, letting $s = \lfloor (j - 2)/2 \rfloor$, one can show that

\[
Q_j(m) = \begin{cases} 2^{(1/s)m^s + O(m^{s-1})}, & \text{for } j \geq 4 \text{ even;} \\ 2^{(1/s)m^s \log_2 m + O(m^s)}, & \text{for } j \geq 3 \text{ odd;} 
\end{cases}
\]

just as in the case of $P_j(m)$.

**Lemma 4.9.** For every $j \geq 2$ and $m \geq 3$ we have

\[
q^{(j)}_{Q_j(m)}(n) = \Omega(n \alpha_{m+1}^{(j-2)}(n))
\]

(where the implicit constants might depend on both $m$ and $j$).

**Proof sketch.** The case $j = 2$ is trivial, and the case $j = 3$ is given by Lemma 4.4. So let $j \geq 4$. We apply Recurrence 4.8 with the following parameters:

\[
k_1 = Q_{j-1}(m), \quad k_2 = Q_{j-2}(m), \quad k_3 = Q_j(m - 1), \quad k = Q_j(m).
\]

We first handle the case $m = 3$, by induction on $j$. For this, let $t = \log^{(j-1)} n$ and $q = \alpha_3(n)$. Note that the recurrence relation

\[
f(n) \geq \frac{n}{t} f(t) + \frac{n}{q}
\]

for our choice of $t$ and $q$, gives $f(n) = \Omega(n \log \alpha_3(n))$, since $\alpha_3(\log^{(i)} n) = \alpha_3(n) - i$.

Then we handle the general case $m \geq 4$ by induction, setting $t = \alpha_{m-1}^{(j-2)}(n)$ and $q = \alpha_m(n)$. Now the recurrence relation (4.10) has solution $f(n) = \Omega(n \log \alpha_m(n))$. 

\qed
Define \( Q'_j(m) \) for \( j \geq 4, \ m \geq 2 \), by

\[
Q'_2(j) = j; \\
Q'_j(m) = Q_j(m - 1), \quad m \geq 3.
\]

Then, using the fact that \( \alpha_{m-1}(n) = \omega(\alpha_m(n)) \) for \( m \geq 2 \), we conclude by Lemmas 3.1, 4.7, and 4.9 that

\[
z_{j}^{(j)}(m)(n) = \omega(n\alpha_m(n)), \quad \text{for all } j \geq 4, m \geq 2.
\]

This proves the lower bounds in Theorem 1.5.

5 Discussion

The most pressing issue is to close the gap between the bounds \( \Omega(r) \) and \( O(r\alpha(r)) \) for the size of weak \( \frac{1}{r} \)-nets for planar sets in convex position. A worst-case bound of \( \Theta(r\alpha(r)) \) would be a major achievement, since there are no known superlinear lower bounds for weak \( \epsilon \)-nets for any fixed dimension \( d \), even for arbitrary point sets.

Another open question is how tight the bounds are for the case of point sets along the moment curve \( \mu_d \). For example, does \( j \) really have to be quadratic in \( d \) in Lemma 2.2?

It would also be nice to find the exact asymptotic form of \( z_{k}^{(j)}(n) \) for every fixed \( j \) and \( k \).

Our divide-and-conquer approach to the problem of stabbing interval chains with triples \( (j = 3) \) is very similar to the approach of Alon and Schieber [3], for a problem related to offline computation of partial sums in semigroups (see also [8, 16]). In fact, both problems have the same asymptotic bounds. (However, we are not aware of any explicit reduction between the two problems.)

Our bounds for weak \( \epsilon \)-nets and for stabbing interval chains also bear a remarkable similarity to the bounds on \( \lambda_s(n) \), the maximum length of a Davenport–Schinzel sequence of order \( s \) on \( n \) symbols. The upper and lower bounds for \( \lambda_s(n) \), for fixed \( s \geq 4 \), have the form \( n - 2^{\Omega(\alpha(n))} \), where the degree of the polynomial in the exponent depends linearly in \( s \) (see Sharir and Agarwal [15]). We do not see any connection between the two problems, which makes this all the more surprising. Moreover, our bounds are slightly sharper than those for \( \lambda_s(n) \); this makes us believe that similar improvements can be established for the bounds on \( \lambda_s(n) \).

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