Non-attacking queens on a triangle

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Most readers are surely familiar with the problem of placing eight non-attacking queens on a chessboard, and its natural generalization to an $n \times n$ board (see the references at the end of this note). Here we consider an interesting variant of this problem, in which the board is triangular.

We are given a triangular board of side n. A queen on the board can move along a straight line parallel to any of the board's sides (see Figure 1). Our problem is to place on the board as many queens as possible, without any two queens attacking each other.

Obviously no more than n queens can be placed, since no row can contain more than one queen.

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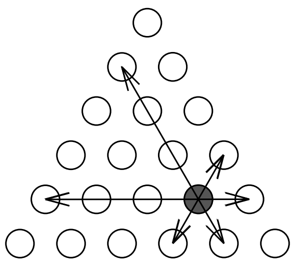

Figure 1: A queen on a triangular board.

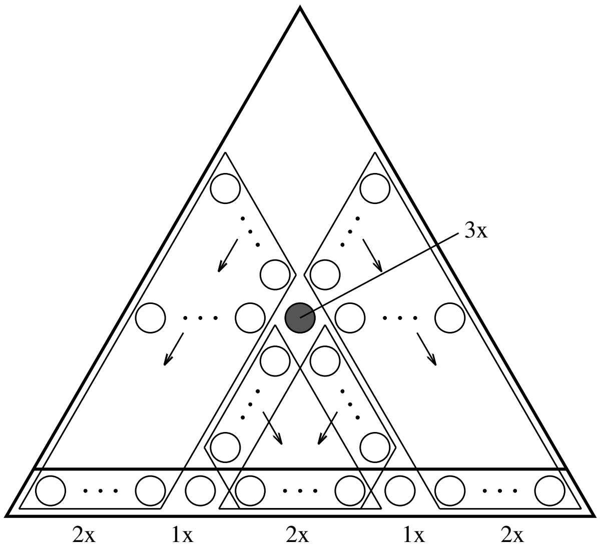

Figure 2: Each queen attacks $2n + 1$ cells.

We now show how to get a tighter bound by counting in two different ways the total number of attacks by queens on cells. Let s be the number of times a cell is attacked by (collinear with) a queen, summed over all the cells of the board. By this definition, a queen attacks its own cell three times—once for each direction of movement.

Each queen contributes exactly $2n+1$ to s, no matter where it is placed. One way to see this is by projecting the cells attacked by a queen onto the board's bottom row, as shown in Figure 2. We have enough cells to cover the bottom row twice, plus one extra cell.

Therefore, if there are q non-attacking queens on the board, then

$$s = (2n + 1)q. \tag{1}$$

On the other hand, each cell can be attacked at most three times. And the number of cells in the board is $1 + 2 + \cdots + n = n(n + 1)/2$, the n-th triangular number. Therefore,

$$s \leq \frac{3n(n + 1)}{2}. \tag{2}$$

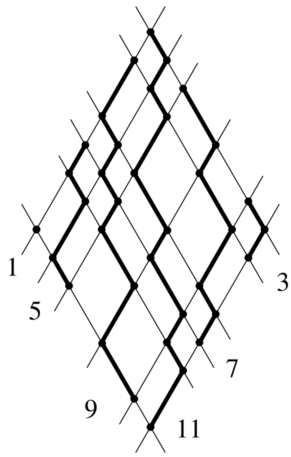

Figure 3: Intersections between two sets of lines, grouped into layers.

Combining (1) and (2), we get

$$q \leq \frac{3n(n+1)}{2(2n+1)} < \frac{3n(n+1)}{4n} = \frac{3(n+1)}{4}.$$

Thus, we cannot place more than about $3n/4$ queens.

We can get a tighter bound by bounding the number of cells that can be actually attacked three times by a given configuration of queens.

Suppose q non-attacking queens are placed on an arbitrarily large board. Trace a line through each queen along each of the three directions of movement. Then, the cells attacked three times are exactly those on which three lines intersect.

Let us ignore the queens for the moment, and concentrate on these three sets of lines, each set containing q parallel lines. Consider two of the sets of lines. The distance between adjacent lines may vary, but the lines will always produce a rhomboid of $q \times q$ intersections, as shown in Figure 3. Group the intersections into "layers" as indicated by the thick segments. The numbers of intersections in the layers will always be $1, 3, 5, \ldots, 2q-1$, the first q odd numbers (as can be easily shown by induction).

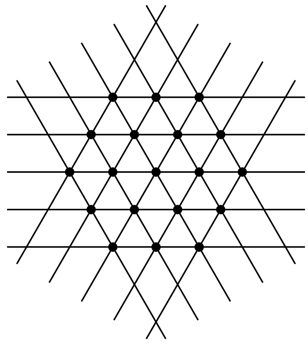

Figure 4: Maximizing the number of triple intersections.

Now, consider the third set of lines (which would be horizontal in the figure). Each such line can cross at most one intersection per layer, since no layer contains two horizontally-aligned intersections.

Therefore, to maximize the number of triple intersections, the most we can do is place the horizontal lines greedily one by one, each one passing through as many of the available intersections as possible. Thus, the first horizontal line can cross at most q intersections; the second line, at most $q-1$; the third line, again at most $q-1$; and so on, until we use up all the q horizontal lines. Figure 4 illustrates how to achieve the maximum number of triple intersections with equidistant lines.

Therefore, the maximum number of triple intersections is, for q even,

$$q + 2\Big((q-1) + (q-2) + \cdots + (\frac{q}{2}+1)\Big) + \frac{q}{2};$$

and for q odd,

$$q + 2\Big((q-1) + (q-2) + \cdots + \frac{q+1}{2}\Big).$$

The above sums are easily calculated in terms of triangular numbers; they equal $3q^2/4$ and $(3q^2+1)/4$, respectively. Thus, for all q, the number of triple intersections is no more than $(3q^2+1)/4$.

Now let us come back to our non-attacking queens on a triangle. It follows that s, the total number of times a cell is attacked by a queen, is bounded by

$$s \leq \frac{2n(n+1)}{2} + \frac{3q^2+1}{4}. \tag{3}$$

(We add 2 for each cell of the board, and then 1 for each cell that can be attacked a third time.)

Combining (1) and (3), we get a quadratic inequality for q. Surprisingly, the solution to this inequality does not involve radicals; the solution is

$$q \leq \frac{2n+1}{3} \quad \text{or} \quad q \geq 2n+1.$$

Obviously, q cannot be larger than n, and q must be an integer. Therefore,

$$q \leq \left\lfloor \frac{2n+1}{3} \right\rfloor. \tag{4}$$

An optimal solution We end by showing that the bound given by equation (4) is in fact tight. Figure 5 illustrates how to place $(2n+1)/3$ queens on a board of side n, when $n \equiv 1 \pmod 3$. If we number the board's columns 1 through $2n-1$ as shown, then the queens are placed on all cells in columns number $(2n+1)/3$ and $(4n+2)/3$.

For the cases $n \equiv 0$ and $n \equiv 2 \pmod 3$, we can use the same configuration and remove the board's bottom one or two rows, respectively, along with their queens. Therefore, we can always place $\lfloor (2n+1)/3 \rfloor$ queens on a board of side n.

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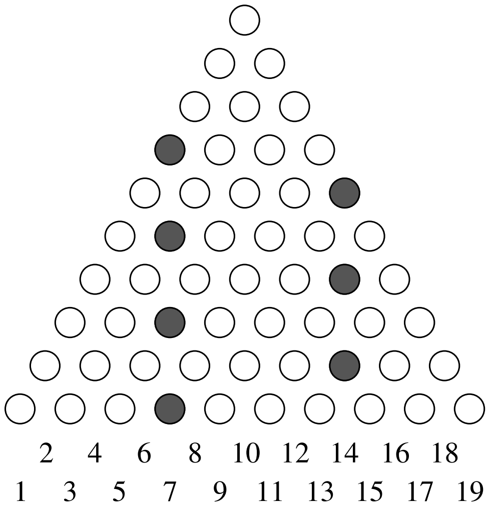

Figure 5: $(2n+1)/3 = 7$ non-attacking queens on a board of side $n = 10$.

References

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