

Lower bounds for weak epsilon-nets and stair-convexity

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Abstract

A set $N \subset \mathbb{R}^d$ is called a *weak ε -net* (with respect to convex sets) for a finite $X \subset \mathbb{R}^d$ if N intersects every convex set C with $|X \cap C| \geq \varepsilon|X|$. For every fixed $d \geq 2$ and every $r \geq 1$ we construct sets $X \subset \mathbb{R}^d$ for which every weak $\frac{1}{r}$ -net has at least $\Omega(r \log^{d-1} r)$ points; this is the first superlinear lower bound for weak ε -nets in a fixed dimension.

The construction is a *stretched grid*, i.e., the Cartesian product of d suitable fast-growing finite sequences, and convexity in this grid can be analyzed using *stair-convexity*, a new variant of the usual notion of convexity.

We also consider weak ε -nets for the diagonal of our stretched grid in \mathbb{R}^d , $d \geq 3$, which is an “intrinsically 1-dimensional” point set. In this case we exhibit slightly superlinear lower bounds (involving the inverse Ackermann function), showing that upper bounds by Alon, Kaplan, Nivasch, Sharir, and Smorodinsky (2008) are not far from the truth in the worst case.

Using the stretched grid we also improve the known upper bound for the so-called *second selection lemma* in the plane by a logarithmic factor: We obtain a set T of t triangles with vertices in an n -point set in the plane such that no point is contained in more than $O(t^2/(n^3 \log \frac{n^3}{t}))$ triangles of T .

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1 Introduction and statement of results

Weak ε -nets. Let $X \subset \mathbb{R}^d$ be a finite set. A set $N \subseteq \mathbb{R}^d$ is called a *weak ε -net*⁴ for X , where $\varepsilon \in (0, 1]$ is a real number, if N intersects every convex set C with $|X \cap C| \geq \varepsilon|X|$. This important notion was introduced by Haussler and Welzl [10] and later applied in results in discrete geometry, most notably in the spectacular proof of the Hadwiger–Debrunner (p, q) -conjecture by Alon and Kleitman [4]. We refer to Matoušek [12, Chap. 10] for a wider background on weak ε -nets and the related notion of (strong) ε -nets, and to Alon, Kálai, Matoušek, and Meshulam [2] for a study of weak ε -nets in an abstract setting.

In this paper, instead of ε , we will often use the parameter $r := \frac{1}{\varepsilon} \geq 1$ and thus speak of weak $\frac{1}{r}$ -nets.

Let $f(X, r)$ denote the minimum cardinality of a weak $\frac{1}{r}$ -net for X . It is a nontrivial fact, first proved by Alon, Bárány, Füredi, and Kleitman [1], that

$$f(d, r) := \sup\{f(X, r) : X \subset \mathbb{R}^d \text{ finite}\}$$

is finite for every $d \geq 1$ and every $r \geq 1$; that is, for every set X there exist weak ε -nets of size bounded solely in terms of d and ε .

Several papers were devoted to estimating the order of magnitude of $f(d, r)$. For $d = 2$, the best upper bound is $f(2, r) = O(r^2)$ [1] (also see [8] for another proof), and for every fixed $d \geq 3$ it is known that $f(d, r) = O(r^d (\log r)^{c_d})$ for some constant c_d (Chazelle, Edelsbrunner, Grigni, Guibas, Sharir, and Welzl [8]; also see Matoušek and Wagner [14] for a simpler proof).

The only known nontrivial lower bound for $f(d, r)$ asserts that $f(d, 50) = \Omega(\exp(\sqrt{d/2}))$ [13]. It concerns the dependence of $f(d, r)$ on d , and no lower bound, except for the obvious estimate $f(d, r) \geq r$, has been known for d fixed and r large. Our main result is a superlinear lower bound for every fixed d .

Theorem 1.1. *Let $d \geq 2$ be fixed. Then for every $r \geq 1$ there exists a finite set $G_s \subset \mathbb{R}^d$ (a stretched grid) such that*

$$f(G_s, r) = \Omega(r \log^{d-1} r).$$

The stretched grid. The stretched grid G_s in the theorem is the Cartesian product $X_1 \times X_2 \times \dots \times X_d$, where each X_i is a suitable set of m real numbers. The integer m is a parameter of the construction of G_s , so we sometimes write $G_s = G_s(m)$, and m has to be chosen sufficiently large in terms of r and d in the proof of Theorem 1.1.

The main idea in the construction of G_s is that X_2, X_3, \dots, X_d are “fast-growing” sequences, and each X_i grows much faster than X_{i-1} . For technical reasons, we will not define $G_s(m)$ uniquely; rather, we will introduce some condition that the X_i have to satisfy, and thus, formally speaking, $G_s(m)$ will stand for a whole class of sets. To simplify calculations, we will also require X_1 to grow quickly.

We will define the X_i by induction on i , together with relations \ll_i on \mathbb{R} , which describe “at least how fast” the terms in X_i must grow (but we will also use \ll_i for comparing real numbers other than the members of X_i). Let us write $X_i = \{x_{i1}, x_{i2}, \dots, x_{im}\}$, where $x_{i1} < x_{i2} < \dots < x_{im}$.

We start by letting $x \ll_1 y$ to mean $K_1 x \leq y$, where $K_1 = 2^d$. Then we choose X_1 so that $x_{11} = 1$ and $x_{11} \ll_1 x_{12} \ll_1 \dots \ll_1 x_{1m}$.

⁴More precisely we should say “weak ε -net for X with respect to convex sets”, since later on we will also consider ε -nets with respect to another family of sets. But since weak ε -nets w.r.t. convex sets are our main object of interest, we reserve the term “weak ε -net” without further specifications for this particular case.

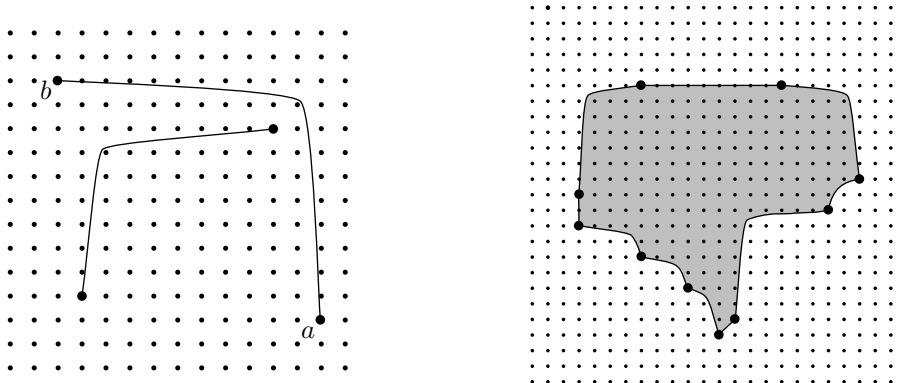


Figure 1: The bijection transforming the stretched grid to the uniform grid: the images of two straight segments connecting grid points (left), and the image of a convex set—the convex hull of the points marked bold (right).

Having defined X_{i-1} and \ll_{i-1} , we set $K_i := 2^d x_{(i-1)m}$, we define $x \ll_i y$ to mean $K_i x \leq y$, and we choose X_i so that $x_{i1} = 1$ and $x_{i1} \ll_i x_{i2} \ll_i \cdots \ll_i x_{im}$.

This construction develops further an idea from our earlier paper [7]. As we will explain, the intersections of convex sets with the stretched grid can be approximated, up to a small error, by sets that have a simple, essentially combinatorial description.

It is practically impossible to make a realistic drawing of the stretched grid, but we can conveniently think about it using a bijection with a uniform (equally spaced) grid. Namely, we define the *uniform grid* in the unit cube $[0, 1]^d$ by

$$G_u = G_u(m) := \left\{0, \frac{1}{m-1}, \frac{2}{m-1}, \dots, \frac{m-1}{m-1}\right\}^d.$$

Let $\text{BB}(G_s) := [1, x_{1m}] \times [1, x_{2m}] \times \cdots \times [1, x_{dm}]$ be the bounding box of G_s , and let $\pi: \text{BB}(G_s) \rightarrow [0, 1]^d$ be a bijection that maps G_s onto G_u and preserves ordering in each coordinate (that is, we map points of G_s to the corresponding points of G_u and we squeeze the “elementary boxes” of G_s onto the corresponding elementary boxes of G_u).

Fig. 1 shows, for $d = 2$, the image under π of two straight segments connecting grid points (left) and of a “generic” convex set (right). The image of the straight segment ab , for example, first ascends almost vertically almost to the level of b , and then it continues almost horizontally towards b . This motivates the following notions.

Stair-convexity. First we define, for points $a = (a_1, a_2, \dots, a_d)$ and $b = (b_1, b_2, \dots, b_d) \in \mathbb{R}^d$, the *stair-path* $\sigma(a, b)$. It is a polygonal path connecting a and b and consisting of at most d closed line segments, each parallel to one of the coordinate axes. The definition goes by induction on d ; for $d = 1$, $\sigma(a, b)$ is simply the segment ab . For $d \geq 2$, after possibly renaming a and b , let us assume $a_d \leq b_d$. We set $a' := (a_1, a_2, \dots, a_{d-1}, b_d)$ and we let $\sigma(a, b)$ be the union of the segment aa' and of the stair-path $\sigma(a', b)$; for the latter we use the recursive definition after “forgetting” the (common) last coordinate of a' and b . See Fig. 2 for examples.

Now we define a set $S \subseteq \mathbb{R}^d$ to be *stair-convex* if for every $a, b \in S$ we have $\sigma(a, b) \subseteq S$. See Fig. 2 again.⁵

⁵Readers familiar with abstract convex spaces might notice that a d -fold cone over the one-element convex structure

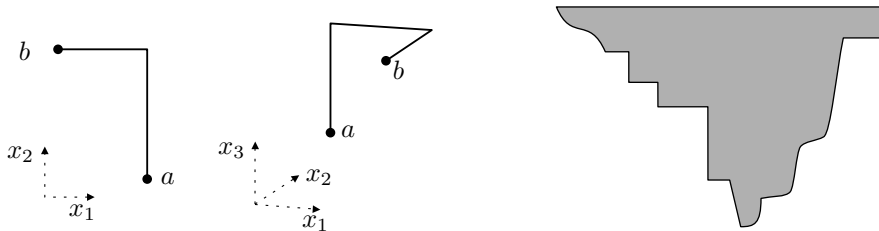


Figure 2: Examples of a stair-path in the plane (left) and in 3-space (center). An example of a stair-convex set in the plane (right).

Since the intersection of stair-convex sets is obviously stair-convex, we can also define the *stair-convex hull* $\text{stconv}(X)$ of a set $X \subseteq \mathbb{R}^d$ as the intersection of all stair-convex sets containing X .

As Fig. 1 indicates, convex sets in the stretched grid transform to “almost” stair-convex sets. We will now express this connection formally.

Epsilon-nets for stair-convex sets and a transference lemma. Let us call a set $N \subseteq [0, 1]^d$ an ε -net for $[0, 1]^d$ with respect to stair-convex sets⁶ if $N \cap S \neq \emptyset$ for every stair-convex $S \subseteq [0, 1]^d$ with $\text{vol}(S) \geq \varepsilon$ (where $\text{vol}(\cdot)$ denotes the d -dimensional Lebesgue measure on $[0, 1]^d$).

Lemma 1.2 (Transference for weak ε -nets).

- (i) Let N be a weak ε -net (w.r.t. convex sets) for the d -dimensional stretched grid $G_s = G_s(m)$ of side m . Then the set $\pi(N) \subseteq [0, 1]^d$ is an ε' -net for $[0, 1]^d$ w.r.t. stair-convex sets with $\varepsilon' \leq \varepsilon + O(|N|/m)$ (with the constant of proportionality depending on d).
- (ii) Let N be an ε -net for $[0, 1]^d$ w.r.t. stair-convex sets. Then $\pi^{-1}(N)$ is a weak ε' -net (w.r.t. convex sets) for $G_s(m)$ with $\varepsilon' \leq \varepsilon + O(|N|/m)$, again with the constant of proportionality depending on d .

The proof is based mainly on the next two lemmas, which will be useful elsewhere as well. The first one is a local characterization of the stair-convex hull.

Let $a = (a_1, \dots, a_d)$ be a point in \mathbb{R}^d . We say that another point $b = (b_1, \dots, b_d) \in \mathbb{R}^d$ has *type 0 with respect to a* if $b_i \leq a_i$ for every $i = 1, 2, \dots, d$. For $j \in \{1, 2, \dots, d\}$ we say that b has *type j with respect to a* if $b_j \geq a_j$ but $b_i \leq a_i$ for all $i = j + 1, \dots, d$. (It may happen that b has more than one type with respect to a , but only if some of the above inequalities are equalities.)

Lemma 1.3. Let $X \subseteq \mathbb{R}^d$ be a point set, and let $x \in \mathbb{R}^d$ be a point. Then $x \in \text{stconv } X$ if and only if X contains a point of type j with respect to x for every $j = 0, 1, \dots, d$.

is almost the same as the family of stair-convex subsets of $[0, 1]^d$. See Van de Vel [18, p. 32] for the definitions. We also note that any line parallel to a coordinate axis intersects a stair-convex set in a (possibly empty) segment. Sets with this latter property are called *rectilinearly convex*, *orthoconvex*, or *separately convex* in various sources. However, stair-convexity is a considerably stronger property.

⁶In order to put this notion, as well as weak ε -nets introduced earlier, into a wider context, we recall the following general definitions, essentially due to Haussler and Welzl. Let Y be a set, let $\mathcal{F} \subseteq 2^Y$ be a system of subsets of Y , and let μ be a finite measure on Y such that all $F \in \mathcal{F}$ are measurable. A set $N \subseteq Y$ is a *weak ε -net for (Y, \mathcal{F}) w.r.t. μ* if $N \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ with $\mu(F) \geq \varepsilon\mu(Y)$. It is an ε -net for (Y, \mathcal{F}) w.r.t. μ if, moreover, N is contained in the support of μ .

The next lemma shows that convex hulls and stair-convex hulls almost coincide in the stretched grid. Let us say that two points $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d)$ in $\text{BB}(G_s)$ are *far apart* if, for every $i = 1, 2, \dots, d$, we have either $a_i \ll_i b_i$ or $b_i \ll_i a_i$. We also extend this notion to sets; $P, Q \subseteq \mathbb{R}^d$ are far apart if each $p \in P$ is far apart from each $q \in Q$.

Lemma 1.4. *Let P and Q be sets in $\text{BB}(G_s)$ that are far apart. Then $\text{stconv}(P) \cap \text{stconv}(Q) \neq \emptyset$ if and only if $\text{conv}(P) \cap \text{conv}(Q) \neq \emptyset$.*

In this paper we use Lemma 1.4 only with $|Q| = 1$ (then it is a statement about membership of a point q in $\text{conv}(P)$). We believe, however, that the above more general version is interesting in its own right and potentially useful in further applications, and thus worth expending some extra effort in the proof. The lemma generalizes a result of [7], but the proof method is different.

Lemmas 1.2–1.4 are proved in Appendix A.

Theorem 1.1 immediately follows from Lemma 1.2(i) and the next proposition:

Proposition 1.5. *Any $\frac{1}{r}$ -net for $[0, 1]^d$ w.r.t. stair-convex sets has at least $\Omega(r \log^{d-1} r)$ points.*

The proof, which we present in Section 2, is strongly inspired by Roth’s beautiful lower bound in discrepancy theory [17]; also see [11] for a presentation of Roth’s proof and a wider context.

As we will show in Appendix B, the lower bound in the proposition is actually tight (up to a constant factor). This means, via Lemma 1.2(ii), that the stretched grid itself is not going to provide any stronger lower bounds for weak ε -nets than those proved here.

Weak ε -nets for “1-dimensional” sets. The smallest possible size of weak ε -nets has also been investigated for special classes of sets [8, 6, 14, 3].

For us, two results of Alon, Kaplan, Nivasch, Sharir, and Smorodinsky [3] are particularly relevant. First, improving on earlier results by Chazelle et al. [8], they proved that for every planar finite set X in convex position we have $f(X, r) = O(r\alpha(r))$, where α denotes the inverse Ackermann function (we recall that $f(X, r)$ is the smallest possible size of a weak $\frac{1}{r}$ -net for X). This, together with our Theorem 1.1, shows that the worst case for weak ε -nets in the plane does *not* occur for sets in convex position.

Second, Alon et al. [3], improving on [14], also showed that if γ is a curve in \mathbb{R}^d that intersects every hyperplane in at most k points, where d and $k \geq d$ are considered constant, then every finite $X \subset \gamma$ has weak $\frac{1}{r}$ -nets of size almost linear in r . We won’t recall the precise formulas, which are somewhat complicated; we just state that the size can be bounded by $r \cdot 2^{C\alpha(r)^b}$, where C and b depend only on d and k .

We will show that for $d \geq 3$, point sets on a curve γ as above (with $k = d$) indeed require weak $\frac{1}{r}$ -nets of size superlinear in r in the worst case, and the form of our lower bound is actually similar to the just mentioned upper bounds, only with smaller values of b .

This time the point set is the *diagonal* D_s of the d -dimensional stretched grid G_s . That is, with $G_s(n) = X_1 \times \dots \times X_d$, where $X_i = \{x_{i1}, \dots, x_{in}\}$, we set $D_s(n) := \{(x_{1j}, \dots, x_{dj}) : j = 1, 2, \dots, n\}$ (this set appeared already in [7], although there it was defined slightly differently).

Theorem 1.6. *For $d \geq 3$ fixed, let us put $t := \lfloor d/2 \rfloor - 1$, and let us define a function β_d by*

$$\beta_d(r) := \begin{cases} \frac{1}{t!} \alpha(r)^t & \text{for } d \text{ even;} \\ \frac{1}{t!} \alpha(r)^t \log_2 \alpha(r) & \text{for } d \text{ odd.} \end{cases}$$

(i) (Lower bound) For every $r \geq 1$ there exists $n_0 = n_0(r)$ such that for all $n \geq n_0$

$$f(D_s(n), r) \geq r \cdot 2^{(1-o(1))\beta_d(r)},$$

where $o(\cdot)$ refers to $r \rightarrow \infty$ and the $o(1)$ term has the form $O(\alpha(r)^{-1})$ for d even and $O((\log_2 \alpha(r))^{-1})$ for d odd. (In particular, for $d = 3$ the lower bound is $\Omega(r\alpha(r))$.)

(ii) (Upper bound) The lower bound from (i) is tight in the worst case up to the $o(1)$ term in the exponent. That is, $f(D_s(n), r) \leq r \cdot 2^{(1+o(1))\beta_d(r)}$, with the same form of the $o(1)$ term as in (i).

This theorem is proved in Appendix C; the proof relies essentially on tools from [3]. In that appendix we will also check that, with a suitable choice of the stretched grid, the set D_s is contained in a curve intersecting every hyperplane at most d times.

We find it quite fascinating that the bounds in the theorem are also identical to the current best upper bounds for a seemingly unrelated problem: the maximum possible length of *Davenport–Schinzel sequences* [15].

“Thin” sets of triangles. Let X be an n -point set in the plane, and let T be a family of t triangles with vertices at the points of X . Bárány, Füredi, and Lovász [5] were the first to prove a statement of the following kind: If T has “many” triangles, then there is a point contained in a “considerable number” of triangles of T . (This kind of statement is called a *second selection lemma* in [12]. Bárány et al. used it in their proof of the first nontrivial upper bound in the so-called *k-set problem* in dimension 3, and their work inspired many further exciting results such as the *colored Tverberg theorem*; see, e.g., [12] for background.) The current best quantitative version is this: There exists a point contained in at least $\Omega(t^3/(n^6 \log^2 n))$ triangles of T (Nivasch and Sharir [16], fixing a proof of Eppstein [9]).

It is not hard to see that this lower bound cannot be improved beyond $O(t^2/n^3)$. Indeed Eppstein [9] showed that for every n -point set $X \subset \mathbb{R}^2$ and for all t between n^2 and $\binom{n}{3}$ there is a set of t triangles with vertices in X such that no point lies in more than $O(t^2/n^3)$ triangles of T .

Here we provide the first (slight) improvement of this easy bound, again using the stretched grid.

Theorem 1.7. *Let $n = m^2$. Then for all t ranging from $n^{2.5} \log n$ to $\binom{n}{3}$ there exists a set of t triangles on the stretched grid $G_s(m)$ such that no point lies in more than*

$$O\left(\frac{t^2}{n^3 \log(n^3/t)}\right)$$

triangles of T . (In particular, if $t < n^{3-\delta}$ for some constant $\delta > 0$, then the bound is $O(t^2/(n^3 \log n))$.)

This theorem is proved in Section 3.

2 The lower bound for stair-convex sets

Here we prove Proposition 1.5, stating that any $\frac{1}{r}$ -net for $[0, 1]^d$ w.r.t. stair-convex sets has $\Omega(r \log^{d-1} r)$ points. Thus, for an arbitrary set $N \subseteq [0, 1]^d$ of n points, it suffices to exhibit a stair-convex set $S \subseteq [0, 1]^d$ of volume at least $\Omega((\log^{d-1} n)/n)$ that avoids N .

We will produce such an S as a union of suitable axis-parallel boxes.

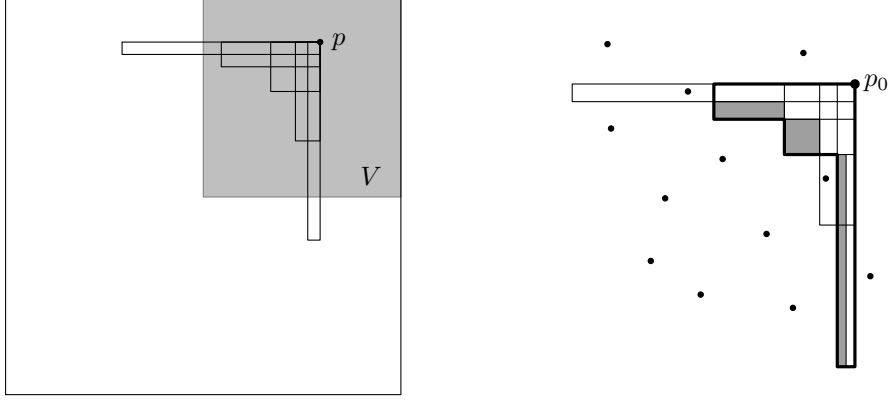


Figure 3: The fan \mathcal{F}_p (left); the stair-convex set S made of the empty boxes of \mathcal{F}_{p_0} and the lower subboxes witnessing the volume of S (right).

Let $k = \Theta(\log n)$ be the integer with $2^{d+1}n \leq 2^k < 2^{d+2}n$, and let us call every integer vector $t = (t_1, t_2, \dots, t_d)$ with $t_i \geq 1$ for all i and with $t_1 + t_2 + \dots + t_d = k$ a *box type*. For later use we record that the number T of box types is $\binom{k-1}{d-1} = \Omega(k^{d-1})$.

Let $V := [\frac{1}{2}, 1]^d$ be the “upper right part” of the cube $[0, 1]^d$. For a box type t and a point $p \in V$, we define the *normal box of type t anchored at p* as

$$B_t(p) := [p_1 - 2^{-t_1}, p_1] \times [p_2 - 2^{-t_2}, p_2] \times \dots \times [p_d - 2^{-t_d}, p_d].$$

Since each side of $B_t(p)$ is at most $\frac{1}{2}$, each normal box is contained in $[0, 1]^d$.

The volume of each normal box is $2^{-k} \leq 1/(2^{d+1}n)$. Let us call a normal box $B_t(p)$ *empty* if $B_t(p) \cap N = \emptyset$. We will show that for every box type t and for $p \in V$ chosen uniformly at random, we have

$$\Pr[B_t(p) \text{ is empty}] \geq \frac{1}{2}. \quad (1)$$

Indeed, for every point $x \in [0, 1]^d$ we have $\text{vol}\{p \in V : x \in B_t(p)\} \leq 2^{-k}$, which in probabilistic terms means $\Pr[x \in B_t(p)] \leq 2^{-k} / \text{vol}(V) = 2^{-k+d} \leq \frac{1}{2n}$, and (1) follows by the union bound.

Now we define the *fan* $\mathcal{F}(p)$ of a point $p \in V$ as the set consisting of the normal boxes $B_t(p)$ for all the T possible box types t (see Fig. 3 left). By (1) we get that for a random $p \in V$ the expected number of empty boxes in the fan of p is at least $T/2$.

Thus, there exists a particular point $p_0 \in V$ such that $\mathcal{F}(p_0)$ has at least $T/2$ empty boxes. We define S as the union of these empty boxes. Then $S \cap N = \emptyset$, S is clearly stair-convex, and it remains to bound from below the volume of S .

For an axis-parallel box $B = [a_1, a_1 + s_1] \times \dots \times [a_d, a_d + s_d]$ we define the *lower subbox* $B' := [a_1, a_1 + \frac{1}{2}s_1] \times \dots \times [a_d, a_d + \frac{1}{2}s_d]$. We observe that if $B_{t_1}(p)$ and $B_{t_2}(p)$ are two normal boxes of different types anchored at the same point, then their lower subboxes are disjoint. Hence, $\text{vol}(S)$ is at least the sum of volumes of the lower subboxes of $T/2$ normal boxes, and so $\text{vol}(S) \geq \frac{T}{2} 2^{-d} 2^{-k} = \Omega((\log^{d-1} n)/n)$. Proposition 1.5 is proved. \square

3 The upper bound for the second selection lemma

Proof of Theorem 1.7. We consider $n = m^2$ and the planar stretched grid $G_s(m)$. Let us now write $G_s(m) = \{x_1, \dots, x_m\} \times \{y_1, \dots, y_m\}$. We want to define a set T of t triangles with vertices

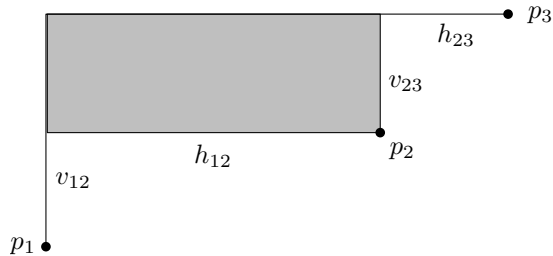


Figure 4: The stair-convex hull of the vertex set of a triangle in $T(h_{12}, h_{23}, v_{12}, v_{23})$.

in $G_s(m)$ that is “thin”, i.e., no point is contained in too many triangles.

Let $\rho \in (0, 1]$ be a parameter, which we will later determine in terms of n and t .

Let $p_1 = (x_{i_1}, y_{j_1})$, $p_2 = (x_{i_2}, y_{j_2})$, $p_3 = (x_{i_3}, y_{j_3})$ be three distinct points of $G_s(m)$. Let us call the triangle $\Delta = p_1 p_2 p_3$ *increasing* if $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$. Let us define the *horizontal dimensions* of Δ as $h_{12} := i_2 - i_1$ and $h_{23} := i_3 - i_2$, and the *vertical dimensions* as $v_{12} := j_2 - j_1$ and $v_{23} := j_3 - j_2$.

We define T as the set of all increasing triangles Δ as above that satisfy

$$\frac{1}{3}m \leq i_2, j_2 \leq \frac{2}{3}m; \quad h_{12}, h_{23}, v_{12}, v_{23} \leq \frac{1}{3}m; \quad h_{12}v_{23} \leq \rho n.$$

The last condition may look mysterious but it will be explained soon. However, first we bound $|T|$ from below, which is routine.

An increasing triangle Δ is determined by p_2 and by its horizontal and vertical dimensions. Each of i_2, j_2, h_{23}, v_{12} can be chosen independently in $\frac{m}{3}$ ways. The pair (h_{12}, v_{23}) can then be chosen, independent of the previous choices, as a lattice point lying in the square $[0, \frac{m}{3}]^2$ and below the hyperbola $xy = \rho n$, and one can easily calculate (by integration, say) that the number of choices is of order $\rho n \log \frac{1}{\rho}$. Thus $|T| = \Omega(n^3 \rho \log \frac{1}{\rho})$, and thus for $\rho := Ct/(n^3 \log(n^3/t))$ with a sufficiently large constant C we obtain $|T| \geq t$ as needed. (Actually, the above calculation of integer points under the hyperbola is valid only if ρ is not too small compared to m , but the assumptions of the theorem and our choice of ρ guarantee $\rho = \Omega(\frac{1}{m})$.)

Let us fix an arbitrary point q in the plane. It remains to bound from above the number of triangles $\Delta \in T$ containing q . To this end, we partition the triangles in T into classes according to their horizontal and vertical dimensions; let $T(h_{12}, h_{23}, v_{12}, v_{23})$ be one of these classes. The total number of triangles in such an equivalence class equals the number of choices of p_2 , so it is $\Theta(n)$. We want to show that only $O(\rho n)$ of them contain q .

We use Lemma 1.4 with $P = \{p_1, p_2, p_3\}$ and $Q = \{q\}$. Then, $q \in \Delta$ may hold only if $q \in \text{stconv}\{p_1, p_2, p_3\}$ or if q is not far apart from at least one of p_1, p_2, p_3 .

If, say, p_2 is not far apart from q , then its position is restricted to two rows or two columns of the grid, and similarly for p_1 and p_3 . Thus, there are only $O(m)$ choices for Δ .

It remains to deal with the case $q \in \text{stconv}\{p_1, p_2, p_3\}$. The stair-convex hull of the vertex set of a triangle $\Delta \in T(h_{12}, h_{23}, v_{12}, v_{23})$ is depicted in Fig. 4 (the picture actually shows the image under π in the uniform grid). It contains $h_{12}v_{23} + O(m) \leq \rho n + O(m)$ grid points, and thus there are at most $\rho n + O(m) = O(\rho n)$ placements of p_2 such that the considered stair-convex hull contains q .

So in every equivalence class of the triangles of T only an $O(\rho)$ fraction of triangles contain q . Thus q lies in no more than $O(\rho|T|) = O(t^2/(n^3 \log(n^3/t)))$ triangles of T as claimed. \square

Remark: A related problem calls for constructing a set of t triangles spanned by n points in \mathbb{R}^3 , such that no *line* in \mathbb{R}^3 stabs too many triangles. The above upper bound does *not* generalize to this

latter problem. This fact gives more weight to our conjecture [7] that the latter, three-dimensional problem has a larger bound than the planar problem.

The first selection lemma and generalizations. In [7] we gave an improved upper bound for the so-called *first selection lemma*, by constructing an n -point set X in \mathbb{R}^d such that no point in \mathbb{R}^d is contained in more than $\left(\frac{n}{d+1}\right)^{d+1} + O(n^d)$ of the d -dimensional simplices spanned by X . The construction was precisely the “main diagonal” D_s of the stretched grid G_s .

Now this can be regarded as a special case of the following result:

Proposition 3.1. *Let $X \subset \mathbb{R}^d$ be an n -point subset of $G_s(m)$ for some m such that every hyperplane perpendicular to a coordinate axis contains only $o(n)$ points of X . (In particular, X can be G_s itself.) Then no point $q \in \mathbb{R}^d$ is contained in more than $(1 + o(1))\left(\frac{n}{d+1}\right)^{d+1}$ of the d -simplices with vertices in X .*

Proof. This follows immediately from the arithmetic-geometric mean inequality since, by Lemmas 1.3 and 1.4, every simplex that contains q (except for at most $o(n^{d+1})$ simplices that have a vertex not lying far apart from q) must have one vertex of each type with respect to q . \square

On a related topic, our calculations suggest that in dimension 3, if we let $X := G_s(\sqrt[3]{n})$, then no line in \mathbb{R}^3 intersects more than $n^3/25 + O(n^2)$ triangles spanned by X . This would prove tightness of another result in [7]. Unfortunately, the calculation, although essentially straightforward, is rather tedious and does not seem to generalize easily. (We would like to find, for general d , j , and k , the maximum number of j -simplices spanned by points of the d -dimensional stretched grid that can be stabbed by a k -flat in \mathbb{R}^d .)

4 Conclusion

In this paper we provide superlinear lower bounds for weak $\frac{1}{r}$ -nets, but the gaps between the known lower and upper bounds for weak $\frac{1}{r}$ -nets are still huge. The most significant gaps are: between $\Omega(r \log r)$ and $O(r^2)$ for the general planar case; between $\Omega(r \log^{d-1} r)$ and $O(r^d \text{polylog } r)$ for the general case in \mathbb{R}^d ; and between $\Omega(r)$ and $O(r\alpha(r))$ for planar point sets in convex position.

The point set that allowed us to obtain the superlinear lower bounds, the stretched grid, might be useful for further problems too, especially since problems about convexity in the stretched grid can be recast in purely combinatorial terms. One might ask, to what extent the stretched grid is “special” as far as weak ε -nets are concerned.

On the one hand, it does provide stronger lower bounds than some other sets. Namely, it is easy to show that there exist weak $\frac{1}{r}$ -nets of size $O(r \log r)$ for the uniform distribution in the d -dimensional unit cube $[0, 1]^d$ (or for any sufficiently large finite uniformly distributed set).⁷ Thus for $d \geq 3$ the stretched grids need strictly larger weak $\frac{1}{r}$ -nets than uniformly distributed sets.

On the other hand, we tend to believe that stretched grids are not special in providing superlinear lower bounds: We conjecture that *no* sets in general position in \mathbb{R}^d , $d \geq 3$, admit linear-size weak $\frac{1}{r}$ -nets. (More precisely: For every C there exist r and n_0 , also possibly depending on d , such that $f(X, r) \geq Cr$ for every $X \subset \mathbb{R}^d$ in general position and with at least n_0 points.) This conjecture may be very hard to prove, though, since it would also imply a superlinear lower bound

⁷This is because there exist $\frac{1}{r}$ -nets of size $O(r \log r)$ with respect to ellipsoids, say, and every convex set of volume ε contains an ellipsoid of volume $\Omega(\varepsilon)$ by the L owner–John theorem; see, e.g., [12] for background.

for $\frac{1}{r}$ -nets for geometrically defined set systems of bounded VC dimension, which has been an outstanding problem in discrete geometry for several decades.

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A Properties of stair-convexity and the transference lemma

Let us first introduce some notation. For a real number y let $h(y)$ denote the “horizontal” hyperplane $\{x \in \mathbb{R}^d : x_d = y\}$. For a horizontal hyperplane $h = h(y)$ let $h^+ := \{x \in \mathbb{R}^d : x_d \geq y\}$ be the upper closed half-space bounded by h , and let h^- be the lower closed half-space. For a set $S \subseteq \mathbb{R}^d$ let $S(y) := S \cap h(y)$ be the horizontal slice of S .

For a point $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ let $\bar{x} := (x_1, \dots, x_{d-1})$ be the projection of x into \mathbb{R}^{d-1} , and define \bar{S} for $S \subseteq \mathbb{R}^d$ similarly. For a point $x \in \mathbb{R}^{d-1}$ and a real number x_d , let $x \times x_d := (x_1, \dots, x_{d-1}, x_d)$, with a slight abuse of notation.

We begin with an equivalent, and perhaps somewhat more intuitive, description of stair-convex sets.

Lemma A.1. *A set $S \subseteq \mathbb{R}^d$ is stair-convex if and only if the following two conditions hold:*

(SC1) *For every $y \in \mathbb{R}$, the set $\overline{S(y)}$ is a $(d-1)$ -dimensional stair-convex set.*

(SC2) (Slice-monotonicity) *For every $y_1, y_2 \in \mathbb{R}$ with $y_1 \leq y_2$ and $S(y_2) \neq \emptyset$, we have $\overline{S(y_1)} \subseteq \overline{S(y_2)}$.*

Proof. First let S be stair-convex. Condition (SC1) is clear from the definition of a stair-path. As for (SC2), we need to prove that for every $a = (a_1, \dots, a_d) \in S(y_1)$ the point $a' := (a_1, \dots, a_{d-1}, y_2)$ directly above a lies in $S(y_2)$. But since $S(y_2) \neq \emptyset$, we can fix some $b \in S(y_2)$, and then a' lies on the stair-path $\sigma(a, b)$ and so $a' \in S(y_2)$ indeed.

Conversely, let $S \subseteq \mathbb{R}^d$ satisfy (SC1) and (SC2), and let $a = (a_1, \dots, a_d), b = (b_1, \dots, b_d) \in S$ with $a_d \leq b_d$. Letting $a' := (a_1, \dots, a_{d-1}, b_d)$ be the point directly above a at the height of b as in the definition of the stair-path $\sigma(a, b)$, we have $\sigma(a', b) \subseteq S$ by the stair-convexity of $S(b_d)$ and $aa' \subseteq S$ by (SC2). \square

Lemma A.2. *The stair-convex hull of a set $X \subseteq \mathbb{R}^d$ can be (recursively) characterized as follows: For every horizontal hyperplane $h = h(y)$ that does not lie entirely above X , let X' stand for the vertical projection of $X \cap h^-$ into h . Then $h \cap \text{stconv}(X) = \text{stconv}(X')$ (where $\text{stconv}(X')$ is a stair-convex hull in dimension $d-1$).*

Proof. First we prove the inclusion $\text{stconv}(X') \subseteq h \cap \text{stconv}(X)$. Let us fix a point $x_0 \in X \cap h^+$ (i.e., above h or on it), and let x be an arbitrary point of $X \cap h^-$. Then x' , the vertical projection of x into h , lies on the stair-path $\sigma(x, x_0)$, and thus $X' \subseteq h \cap \text{stconv}(X)$. Since $h \cap \text{stconv}(X)$ is stair-convex (by (SC1) in Lemma A.1), we also have $\text{stconv}(X') \subseteq h \cap \text{stconv}(X)$.

To establish the reverse inclusion, it suffices to show that for every $(d-1)$ -dimensional stair-convex $S' \subseteq h$ that contains X' there is a d -dimensional stair-convex set S with $S \cap h = S'$ that contains X . Such an S can be defined as $(\mathbb{R}^d \setminus h^-) \cup P^-(S')$, where $P^-(S') = \{(x_1, \dots, x_d) \in \mathbb{R}^d : (x_1, \dots, x_{d-1}, y) \in S', x_d \leq y\}$ is the semi-infinite vertical prism obtained by extruding S' downwards. \square

Next, we prove Lemma 1.3, which asserts that a point x lies in the stair-convex hull of a set X if and only if X contains a point of type j with respect to x for every $j = 0, 1, \dots, d$.

Proof of Lemma 1.3. Both directions follow by induction on d . The case $d = 1$ is trivial, and so we assume $d \geq 2$.

Let h be the horizontal hyperplane containing x . First we suppose $x \in \text{stconv} X$. There exists a point $p_d \in X$ whose last coordinate is at least as large as that of x , and this p_d has type d with respect to x .

Next, let X' be the vertical projection of $X \cap h^-$ into h as in Lemma A.2. By that lemma we have $x \in \text{stconv} X'$, and so, by induction, X' contains points p'_0, \dots, p'_{d-1} (not necessarily distinct) of types $0, \dots, d-1$, respectively, with respect to x . The corresponding points $p_0, \dots, p_{d-1} \in X$ also have types $0, \dots, d-1$ with respect to x .

For the other direction, we suppose that there are points $p_0, \dots, p_d \in X$ of types $0, \dots, d$ with respect to x . Then the vertical projections of p_0, \dots, p_{d-1} into h also have types $0, \dots, d-1$ w.r.t. x , and so by the inductive hypothesis, their stair-convex hull contains x . Since $p_d \in h^+$, it follows, again by Lemma A.2, that $x \in \text{stconv}\{p_0, \dots, p_d\}$. \square

The following lemma describes when stair-convex hulls of two sets intersect. It is the last lemma we need before proving Lemma 1.4.

Lemma A.3. *Let $P, Q \subset \mathbb{R}^d$ be two finite, non-empty sets satisfying $p_k \neq q_k$ for every $p \in P, q \in Q$ and every $1 \leq k \leq d$. Let the highest points of P and Q (in the last coordinate) be p_{top} and q_{top} , respectively; let us write $P^* := P \setminus \{p_{\text{top}}\}$, $Q^* := Q \setminus \{q_{\text{top}}\}$; and let us assume that q_{top} lies above p_{top} . Let $s := |P|$ and $t := |Q|$.*

- (a) *If $s + t < d + 2$, then $\text{stconv}(P) \cap \text{stconv}(Q) = \emptyset$.*
- (b) *If $s + t = d + 2$, then $\text{stconv}(P)$ and $\text{stconv}(Q)$ intersect in at most one point. Specifically, if $t \geq 2$ and p_{top} lies above Q^* , then*

$$\text{stconv}(P) \cap \text{stconv}(Q) = \left(\text{stconv}(\overline{P}) \cap \text{stconv}(\overline{Q^*}) \right) \times \{p_{\text{top},d}\} \quad (2)$$

(which might or might not be empty); otherwise, $\text{stconv}(P) \cap \text{stconv}(Q) = \emptyset$.

- (c) *(A Kirchberger-type result) If $s + t > d + 2$ and $\text{stconv}(P) \cap \text{stconv}(Q) \neq \emptyset$, then there are $P_0 \subset P$ and $Q_0 \subset Q$ of total size $|P_0| + |Q_0| = d + 2$ with $\text{stconv}(P_0) \cap \text{stconv}(Q_0) \neq \emptyset$.*

Proof. First we prove (a) and (b) by induction on d . The base case $d = 1$ is trivial (interpreting the right-hand side of (2) as $\{p_{\text{top}}\}$), so let $d \geq 2$. Let us suppose that $\text{stconv}(P) \cap \text{stconv}(Q) \neq \emptyset$ and $s + t \leq d + 2$. Let $r \in \text{stconv}(P) \cap \text{stconv}(Q)$ be an arbitrary point of the intersection.

From $r \in \text{stconv}(P)$ it follows that $r_d \leq p_{\text{top},d}$. If we had $r_d < p_{\text{top},d}$, then Lemma A.2 would imply that $\bar{r} \in \text{stconv}(\overline{P^*}) \cap \text{stconv}(\overline{Q^*})$. However $|P^*| + |Q^*| = s + t - 2 < (d - 1) + 2$, so by the induction hypothesis the intersection should be empty—a contradiction. Thus $r_d = p_{\text{top},d}$.

Similarly, if $t \geq 2$ and p_{top} is below some point $q \in Q^*$, then $\bar{r} \in \text{stconv}(\overline{P}) \cap \text{stconv}(\overline{Q^* \setminus \{q\}})$, again contradicting the induction hypothesis. Moreover, if $t = 1$ then $\text{stconv}(P) \cap \text{stconv}(Q) = \emptyset$ simply because Q lies entirely above P . Hence we can assume that $t \geq 2$, and that p_{top} is on the same height as r and above Q^* . Then from the Lemma A.2 it follows that $r \in \text{stconv}(P) \cap \text{stconv}(Q)$ if and only if $\bar{r} \in \text{stconv}(\overline{P}) \cap \text{stconv}(\overline{Q^*})$. If $s + t < d + 2$, then $|P| + |Q^*| < (d - 1) + 2$, so $\text{stconv}(P) \cap \text{stconv}(Q) = \emptyset$. Finally, if $s + t = d + 2$, then $\text{stconv}(P) \cap \text{stconv}(Q) = \left(\text{stconv}(\overline{P}) \cap \text{stconv}(\overline{Q^*}) \right) \times \{p_{\text{top},d}\}$, as claimed.

The proof of (c) also follows by induction on d . The case $d = 1$ is clear, so let $d \geq 2$. Suppose $s + t > d + 2$, let $r \in \text{stconv}(P) \cap \text{stconv}(Q)$, and let $h = h(r_d)$ be the horizontal hyperplane

containing r . Then, with $P^- := P \cap h^-$ and $Q^- := Q \cap h^-$, we have $\bar{r} \in \text{stconv}(\overline{P^-}) \cap \text{stconv}(\overline{Q^-})$ by Lemma A.2.

By the inductive hypothesis there are subsets $P_0^- \subseteq P^-$ and $Q_0^- \subseteq Q^-$ with $|P_0^-| + |Q_0^-| = d + 1$ and $\text{stconv}(\overline{P_0^-}) \cap \text{stconv}(\overline{Q_0^-}) \neq \emptyset$. Let $x \in \mathbb{R}^{d-1}$ be a point in this intersection (it need not be identical to \bar{r}).

Let a and b be the highest points of P_0^- and Q_0^- , respectively. If a lies below b , we set $P_0 := P_0^- \cup \{p_{\text{top}}\}$ and $Q_0 := Q_0^-$; it is easily checked that $x \times b_d \in \text{stconv}(P_0) \cap \text{stconv}(Q_0)$. Finally, if a lies above b , we set $P_0 := P_0^-$ and $Q_0 := Q_0^- \cup \{q_{\text{top}}\}$; then $x \times a_d \in \text{stconv}(P_0) \cap \text{stconv}(Q_0)$. \square

Proof of lemma 1.4. We consider finite sets $P, Q \subset \text{BB}(G_s)$ that are far apart. Then, in particular, no point of P shares any coordinate with any point of Q (as is assumed in Lemma A.3).

First we prove that $\text{stconv}(P) \cap \text{stconv}(Q) \neq \emptyset$ implies $\text{conv}(P) \cap \text{conv}(Q) \neq \emptyset$. The proof proceeds by induction on d . In the inductive step we discard the point of the largest height, and find an intersection of the convex hulls of the remaining sets, which is a $(d - 1)$ -dimensional situation. Then we would like to use the discarded point for adjusting the last coordinate of the intersection. In order to make this last step work, instead of simply discarding the highest point, we use it to perturb the other points.

In more detail, let $P = \{p_1, \dots, p_s\}$ and $Q = \{q_1, \dots, q_t\}$ satisfy $\text{stconv}(P) \cap \text{stconv}(Q) \neq \emptyset$. Let p_s, q_t be the highest points in P, Q , respectively, and let us assume q_t lies above p_s . By Lemma A.3(c) we may and will assume that $s + t = d + 2$. Further, by Lemma A.3(b), the set $Q^* := \{q_1, \dots, q_{t-1}\}$ lies below p_s .

We show by induction on d that the following system of equations and inequalities with unknowns $a_1, \dots, a_s, b_1, \dots, b_t$ has a solution:

$$a_1 + \dots + a_s = b_1 + \dots + b_t, = 1 \quad (3a)$$

$$a_1 p_1 + \dots + a_s p_s = b_1 q_1 + \dots + b_t q_t, \quad (3b)$$

$$a_1, \dots, a_s, b_1, \dots, b_t \geq 1/x_{dm}, \quad (3c)$$

where, as we recall, x_{dm} is the maximum height of a point in $\text{BB}(G_s)$. Equations (3a) and (3b) assert that $\text{conv}(P) \cap \text{conv}(Q) \neq \emptyset$, and the inequalities (3c) are crucial in the induction argument.

The case $d = 1$ is an easy computation, and so we assume $d \geq 2$.

Let $\alpha > 0$ be a parameter, and for each $q_i \in Q^*$ we define the ‘‘perturbed’’ point $q'_i = (1 - \alpha)q_i + \alpha q_t$, which lies on the segment $q_i q_t$. Let $Q' = \{q'_1, \dots, q'_{t-1}\}$. Since q_t is very high above Q^* , the segments $q_i q_t$ are ‘‘almost’’ vertical. As we will see, it is possible to choose the parameter α large enough so that Q' lies above p_s , and yet small enough so that $\overline{Q'}$ is not ‘‘too far’’ from $\overline{Q^*}$.

Specifically, we will choose $\alpha \in [1/x_{dm}, 1/2x_{(d-1)m}]$. We claim that for any such choice of α , the set $\overline{Q'}$ is far apart from \overline{P} and $\text{stconv}(\overline{Q'})$ intersects $\text{stconv}(\overline{P})$ iff $\text{stconv}(\overline{Q^*})$ does.

To see this, recall that $A \ll_i B$ means $K_i A \leq B$ with $K_i = 2^d x_{(i-1)m}$. To indicate the dependence on d , we temporarily adopt the more verbose notation $A \ll_{i,d} B$ in place of $A \ll_i B$. Since $\alpha \leq 1/2x_{(d-1)m}$, for every $i = 1, 2, \dots, t$ and every $k = 1, 2, \dots, d - 1$ we have $\alpha q_{ik} \leq \frac{1}{2}$, and so

$$\begin{aligned} q_{ik} \ll_{k,d} p_{jk} &\implies q'_{ik} = (1 - \alpha)q_{ik} + \alpha q_{tk} \leq q_{ik} + \frac{1}{2} \leq 2q_{ik} \ll_{k,d-1} p_{jk}, \\ \text{and } q_{ik} \gg_{k,d} p_{jk} &\implies q'_{ik} = (1 - \alpha)q_{ik} + \alpha q_{tk} \geq q_{ik} - \frac{1}{2} \geq \frac{1}{2}q_{ik} \gg_{k,d-1} p_{jk}. \end{aligned} \quad (4)$$

So indeed, by Lemma A.3, there is no combinatorial change between $\overline{Q^*}$ and $\overline{Q'}$ as far as intersection of stair-convex hulls with $\text{stconv}(\overline{P})$ is concerned.

Further, since q_t is the highest point in $P \cup Q$, we have $\text{stconv}(\overline{P}) \cap \text{stconv}(\overline{Q^*}) \neq \emptyset$, and therefore, $\text{stconv}(\overline{P}) \cap \text{stconv}(\overline{Q'}) \neq \emptyset$. Hence, by the induction hypothesis there exists a point $r \in \text{conv}(\overline{P}) \cap \text{conv}(\overline{Q'})$ and further, there exist real numbers $a_1, \dots, a_s, b_1, \dots, b_{t-1}$ satisfying

$$\begin{aligned} a_1 + \dots + a_s &= b_1 + \dots + b_{t-1} = 1, \\ a_1 \overline{p}_1 + \dots + a_s \overline{p}_s &= b_1 \overline{q}'_1 + \dots + b_{t-1} \overline{q}'_{t-1} = r, \\ a_1, \dots, a_s, b_1, \dots, b_{t-1} &\geq 1/x_{(d-1)m}. \end{aligned} \tag{5}$$

Thus, there are real numbers h_P and h_Q for which $r \times h_P \in \text{conv}(P)$ and $r \times h_Q \in \text{conv}(Q)$. Now we exploit the freedom in choosing α . First let $\alpha := 1/2x_{(d-1)m}$; then we have $q'_{id} > \alpha q_{td} = q_{td}/2x_{(d-1)m} \geq p_{sd}$ (since $q_{td} \gg_d p_{sd}$). Thus, Q' lies entirely above p_s , implying that $h_Q > h_P$ in this case.

Next, let $\alpha := 1/x_{dm}$. Since $p_{sd} \gg_d q_{id}$ for every $i = 1, 2, \dots, t-1$, we have $a_s p_{sd} \geq p_{sd}/x_{(d-1)m} > q_{id} + 1 \geq q_{id} + \alpha q_{td} > q'_{id}$. Hence

$$a_1 p_{1d} + \dots + a_s p_{sd} > a_s p_{sd} > b_1 q'_{1d} + \dots + b_{t-1} q'_{(t-1)d},$$

implying that $h_P > h_Q$ in this case.

Since solutions of linear equations depend continuously on the coefficients, the intermediate value theorem implies that there is an α in the interval $[1/x_{dm}, 1/2x_{(d-1)m}]$ for which $h_Q = h_P$. Fix this α . Then the point $r \times h_P = r \times h_Q$ lies in both $\text{conv}(P)$ and $\text{conv}(Q)$, as desired. It remains to verify the inequalities (3c). We have

$$\begin{aligned} r \times h_P &= a_1 p_1 + \dots + a_s p_s, \\ r \times h_Q &= (1 - \alpha) b_1 q_1 + \dots + (1 - \alpha) b_{t-1} q_{t-1} + \alpha q_t. \end{aligned}$$

Then $a_i \geq 1/x_{dm}$ follows from $a_i \geq 1/x_{(d-1)m}$; the inequality $(1 - \alpha) b_i \geq 1/x_{dm}$ follows since $1 - \alpha \geq 1/2$ and by the definition of x_{dm} ; and $\alpha \geq 1/x_{dm}$ holds by our very choice of α . The first implication in Lemma 1.4 is proved.

We now tackle the reverse implication, again proceeding by induction on d . Let us suppose that $\text{conv}(P) \cap \text{conv}(Q) \neq \emptyset$. By Kirchberger's theorem [12, p. 13], we can assume that $|P| + |Q| \leq d + 2$. As above let $P = \{p_1, \dots, p_s\}$, $Q = \{q_1, \dots, q_t\}$, with points p_s and q_t highest in their respective sets, and with q_t higher than p_s .

Let $r \in \text{conv}(P) \cap \text{conv}(Q)$. Then there exist nonnegative coefficients $a_1, \dots, a_s, b_1, \dots, b_{t-1}$, and α that satisfy

$$\begin{aligned} r &= a_1 p_1 + \dots + a_s p_s = b_1 q_1 + \dots + b_{t-1} q_{t-1} + \alpha q_t, \\ a_1 + \dots + a_s &= b_1 + \dots + b_{t-1} + \alpha = 1. \end{aligned}$$

Since $\sum a_i p_{id} \leq p_{sd}$, it follows that $\alpha \leq p_{sd}/q_{td} \leq 1/2x_{(d-1)m}$.

As in the proof of the first implication, let $Q^* := \{q_1, \dots, q_{t-1}\}$, let $q'_i = (1 - \alpha)q_i + \alpha q_t$ (with the α just introduced), and let $Q' = \{q'_1, \dots, q'_{t-1}\}$. Then r is a convex combination of the points in Q' , so $r \in \text{conv}(Q')$.

Therefore, $\text{conv}(\overline{P}) \cap \text{conv}(\overline{Q'}) \neq \emptyset$, and so by the induction hypothesis $\text{stconv}(\overline{P}) \cap \text{stconv}(\overline{Q'}) \neq \emptyset$. But arguing again as in (4), the order of points of Q' with respect to \overline{P} is same as that of Q^* with respect to \overline{P} in each coordinate, so we also have $\text{stconv}(\overline{P}) \cap \text{stconv}(\overline{Q^*}) \neq \emptyset$. Thus, by Lemma A.3(b), for showing that $\text{stconv}(P) \cap \text{stconv}(Q) \neq \emptyset$ it suffices to prove that p_s lies above Q^* .

Suppose it is not the case, and that q_{t-1} , say, lies above p_s . Since $r \in \text{conv}(Q)$, there exists a point q° in the segment $q_{t-1}q_t$ such that $r \in \text{conv}(Q^\circ)$, where $Q^\circ = \{q_1, \dots, q_{t-2}, q^\circ\}$.

Thus, $\text{conv}(P) \cap \text{conv}(Q^\circ) \neq \emptyset$. Since q° lies above p_s , we can apply the preceding argument with Q° in place of Q , and we infer that $\text{stconv}(\overline{P}) \cap \text{stconv}(\overline{\{q_1, \dots, q_{t-2}\}}) \neq \emptyset$. However, these sets have at total of only d points, contradicting Lemma A.3(a). \square

Next, we derive auxiliary results needed for the proof of transference lemma (Lemma 1.2). Given sets $P, Q \subseteq \mathbb{R}^d$, we define the operation

$$P \ominus Q := \{p \in P : p + Q \subseteq P\},$$

where $p + Q = \{p + q : q \in Q\}$.

Lemma A.4. *Let $S \subseteq [0, 1]^d$ be a stair-convex set, and let $G_u = G_u(m)$ be the uniform grid of side m . Then, for every $\delta > 0$, the set $S^{\delta^-} := S \ominus [0, \delta]^d$ is stair-convex,*

$$\text{vol}(S^{\delta^-}) \geq \text{vol}(S) - d\delta, \quad \text{and} \quad |S^{\delta^-} \cap G_u| \geq |S \cap G_u| - d\lceil(m-1)\delta\rceil m^{d-1}.$$

Proof. For an index $i \in \{1, 2, \dots, d\}$ and $\delta > 0$ let $s_i(\delta)$ be the initial closed segment of the positive x_i -axis (starting at the origin) of length δ .

We prove that for every stair-convex $S \subseteq [0, 1]^d$ and every $\delta > 0$ the set $S' := S \ominus s_i(\delta)$ is stair-convex, has volume at least $\text{vol}(S) - \delta$, and contains at least $|S \cap G_u| - \lceil(m-1)\delta\rceil m^{d-1}$ points of G_u . The assertion of the lemma then follows by d -fold application of this statement and by noticing that $S \ominus [0, \delta]^d = S \ominus s_1(\delta) \ominus \dots \ominus s_d(\delta)$.

As for the stair-convexity of S' , the following actually holds: If S is stair-convex and D is arbitrary, then $S \ominus D$ is stair-convex too. This follows from the translation invariance of stair-paths. Namely, $\sigma(a+x, b+x) = x + \sigma(a, b)$, and thus for $a, b \in S \ominus D$ we have $a+x$ and $b+x$ in S for all $x \in D$, so $x + \sigma(a, b) = \sigma(a+x, b+x) \subseteq S$, and thus $\sigma(a, b) \subseteq S \ominus D$.

The claim about $\text{vol}(S')$ follows by Fubini's theorem, since $S \setminus S'$ intersects every line parallel to the x_i -axis in a single segment of length at most δ . The claim about the number of grid points follows similarly, by noticing that the grid $G_u(m)$ has step $\frac{1}{m-1}$ and thus $S \setminus S'$ contains at most $\lceil\delta(m-1)\rceil$ grid points on each line parallel to the x_i -axis. \square

Corollary A.5 (Grid approximation). *Let $S \subseteq [0, 1]^d$ be a stair-convex set, and let $g_S = |S \cap G_u(m)|$ be the number of points of the uniform grid contained in S . Then,*

$$|g_S - (m-1)^d \text{vol}(S)| \leq dm^{d-1}.$$

Proof. Let $\delta := \frac{1}{m-1}$ be the step of the grid $G_u(m)$. For every grid point $p \in G_u \cap S^{\delta^-}$, the cube $p + [0, \delta]^d$ is contained in S , and since such cubes have disjoint interiors, we have $\text{vol}(S) \geq \delta^d |S^{\delta^-} \cap G_u| \geq \delta^d g_S - \delta^d dm^{d-1}$ by the second inequality in Lemma A.4. Multiplying by δ^{-d} we get $A \leq (m-1)^d \text{vol}(S) + dm^{d-1}$, one of the inequalities in the corollary.

For the other inequality, we observe that if $p \in G_u(m)$ is a grid point such that the cube $p + [-\delta, 0]^d$ intersects S^{δ^-} , then $p \in S$. So using the first inequality of Lemma A.4 gives $\text{vol}(S) \leq \text{vol}(S^{\delta^-}) + d\delta \leq \delta^d g_S + d\delta$, and we are done. \square

Proof of Lemma 1.2. Let us prove part (i). So let N a weak ε -net for $G_s = G_s(m)$, and let $s = |N|$.

Let us call a point $p \in G_s$ *good* if it is far apart from every point of N ; otherwise, p is *bad*. There are at most $2dsm^{d-1}$ bad points in G_s .

Let $\varepsilon' := \varepsilon + 2d(s + 1)/m$, and let us consider a stair-convex set $S' \subseteq [0, 1]^d$ of volume ε' . By Corollary A.5, S' contains a set $P' \subseteq G_u$ of at least $\varepsilon'(m - 1)^d - dm^{d-1}$ grid points.

Let $P = \pi^{-1}(P')$ be the corresponding subset of G_s . By removing all bad points from P we obtain a set P^* of at least $\varepsilon'(m - 1)^d - d(2s + 1)m^{d-1} \geq \varepsilon m^d$ good points. Since N is a weak ε -net, there exists a point $x \in N \cap \text{conv } P^*$.

Since all points of P^* are far apart from x , it follows by Lemma 1.4 that $x \in \text{stconv}(P^*)$. Since π preserves order in each coordinate, we have

$$x' := \pi(x) \in \text{stconv}(\pi(P^*)) \subseteq S'.$$

Since $x' \in \pi(N)$, this proves that $\pi(N)$ intersects every stair-convex set of volume ε' in $[0, 1]^d$. This finishes the proof of part (i) of the transference lemma.

Part (ii) is proved similarly, only with the roles of convexity and stair-convexity interchanged. \square

B Constructing ε -nets w.r.t. stair-convex sets

Here we show that Proposition 1.5 is asymptotically the best possible; that is, for every $r \geq 1$ there exists a set $N \subset [0, 1]^d$, $|N| = O(r \log^{d-1} r)$, intersecting every stair-convex $S \subseteq [0, 1]^d$ with $\text{vol}(S) \geq \frac{1}{r}$.

We begin with the following fact: *For every $s \geq 1$ there exists a set $N \subset [0, 1]^d$ of size $O(s)$ intersecting every axis-parallel box $B \subseteq [0, 1]^d$ with $\text{vol}(B) \geq \frac{1}{s}$.* Indeed, the Van der Corput set in the plane and the Halton–Hammersley sets in dimension d have this property, as well as many other constructions of low-discrepancy sets (Faure sets, digital nets of Sobol, Niederreiter and others, etc.); see, e.g., [11].

Given $r \geq 1$, we now set $s := Cr \log^{d-1} r$ for a sufficiently large constant C , and we let N be a set as in the just mentioned fact. We claim that N is the desired $\frac{1}{r}$ -net for $[0, 1]^d$ w.r.t. stair-convex sets. This follows from the next lemma.

Lemma B.1. *Let $S \subseteq [0, 1]^d$ be a stair-convex set that contains no axis-parallel box of volume larger than v , $0 < v \leq 1/e$ (here, $e = 2.71828\dots$). Then $\text{vol}(S) \leq ev \ln^{d-1} \frac{1}{v}$.*

Proof. We proceed by induction on d . The base case $d = 1$ is trivial, so we assume $d \geq 2$.

Without loss of generality we can assume S intersects the “upper facet” of $[0, 1]^d$ (the facet of $[0, 1]^d$ with last coordinate equal to 1).

For $z \in [0, 1]$ let $h = h(1 - z)$ denote the “horizontal” hyperplane $\{x_d = 1 - z\}$. Let $S' := S \cap h$, and let B be an axis-parallel box of maximum $(d - 1)$ -dimensional volume in S' . We have $\text{vol}_{d-1}(B) \leq \frac{v}{z}$, for otherwise, B could be extended upwards into a box of d -dimensional volume larger than v .

Since S' is stair-convex, for $z \geq ev$ the inductive assumption gives $\text{vol}(S') \leq \frac{ev}{z} \ln^{d-2} \frac{z}{v}$. We also have $\text{vol}(S') \leq 1$. So for $v \leq 1/e$ we have

$$\begin{aligned} \text{vol}(S) &\leq \int_0^{ev} dz + \int_{ev}^1 \frac{ev}{z} \ln^{d-2} \frac{z}{v} dz = ev + \frac{ev}{d-1} \left(\ln^{d-1} \frac{1}{v} - 1 \right) \\ &\leq ev + ev \left(\ln^{d-1} \frac{1}{v} - 1 \right) = ev \ln^{d-1} \frac{1}{v}. \end{aligned}$$

This finishes the induction step. \square

C The diagonal of the stretched grid

Here we prove our results on the diagonal $D_s = D_s(n)$ of the stretched grid $G_s(n)$. We start by showing that, if G_s is defined appropriately, then D_s lies on a curve that intersects every hyperplane in at most d points.

Indeed, if each element x_{ij} of each X_i in the definition of G_s is chosen *minimally*, then we have $x_{ij} = K_i^{j-1}$, and so

$$D_s = \{(K_1^t, \dots, K_d^t) : t = 0, 1, \dots, n-1\}.$$

Thus, D_s is a subset of the curve

$$\gamma = \{(K_1^t, \dots, K_d^t) : t \in \mathbb{R}\}.$$

Lemma C.1. *Let $\gamma \subset \mathbb{R}^d$ be a curve of the form*

$$\gamma = \{(c_1^t, \dots, c_d^t) : t \in \mathbb{R}\},$$

for some positive constants c_1, \dots, c_d . Then every hyperplane in \mathbb{R}^d intersects γ at most d times.

Proof. The claim is equivalent to showing that the function

$$f(t) = \alpha_1 c_1^t + \dots + \alpha_d c_d^t + \alpha_{d+1}$$

has at most d zeros for any choice of parameters $\alpha_1, \dots, \alpha_{d+1}$. Letting $\beta_i = \alpha_i \ln c_i$, it suffices to show that

$$f'(t) = \beta_1 c_1^t + \beta_2 c_2^t + \dots + \beta_d c_d^t = c_1^t (\beta_1 + \beta_2 (c_2/c_1)^t + \dots + \beta_d (c_d/c_1)^t)$$

has at most $d-1$ zeros. But c_1^t never equals zero, so the claim follows by induction. \square

Next, we prove Theorem 1.6, the lower and upper bounds for the size of weak $\frac{1}{r}$ -nets for the diagonal D_s of the stretched grid. We reduce the problem to results of Alon et al. [3] concerning the problem of *stabbing interval chains*.

C.1 The Ackermann function and its inverse

We introduce the Ackermann function and its inverse following [15]:

The *Ackermann hierarchy* is a sequence of functions $A_k(n)$, for $k \geq 1$ and $n \geq 0$, where $A_1(n) = 2n$, and for $k \geq 2$ we let $A_k(n) = A_{k-1}^{(n)}(1)$. (Here $f^{(n)}$ denotes the n -fold composition of f .) The definition of $A_k(n)$ for $k \geq 2$ can also be written recursively: $A_k(0) = 1$, and $A_k(n) = A_{k-1}(A_k(n-1))$ for $n \geq 1$. We have $A_2(n) = 2^n$, and $A_3(n) = 2^{2^{\dots^2}}$ is a tower of n twos.

We have $A_k(1) = 2$ and $A_k(2) = 4$, but $A_k(3)$ already grows very rapidly with k . We define the *Ackermann function* as $A(n) = A_n(3)$. Thus, $A(n) = 6, 8, 16, 65536, \dots$ for $n = 1, 2, 3, \dots$

We then define the slow-growing inverses of these rapidly-growing functions as $\alpha_k(x) = \min\{n : A_k(n) \geq x\}$ and $\alpha(x) = \min\{n : A(n) \geq x\}$ for all real $x \geq 0$.

Alternatively, and equivalently, we can define these inverse functions directly: We define the *inverse Ackermann hierarchy* by letting $\alpha_1(x) = \lceil x/2 \rceil$ and, for $k \geq 2$, defining $\alpha_k(x)$ recursively by $\alpha_k(x) = 0$ for $x \leq 1$, and $\alpha_k(x) = 1 + \alpha_k(\alpha_{k-1}(x))$ for $x > 1$. In other words, for each $k \geq 2$, $\alpha_k(x)$ denotes the number of times we must apply α_{k-1} , starting from x , until we reach a value not larger

than 1. Thus, $\alpha_2(x) = \lceil \log_2 x \rceil$, and $\alpha_3(x) = \log^* x$. Finally, we define the *inverse Ackermann function* by $\alpha(x) = \min \{k : \alpha_k(x) \leq 3\}$.

Thus, $\alpha_{\alpha(x)}(x) \leq 3$ by definition. In fact, we have $\alpha_{\alpha(x)}(x) = 3$ for all $x \geq 5$. Since $\alpha_{k+1}(x) \leq \alpha_k(x) - 1$ whenever $\alpha_k(x) \geq 4$, we have

$$\alpha_{\alpha(x)-k}(x) \geq k + 3 \tag{6}$$

for all x with $\alpha(x) \geq k + 1$.⁸

C.2 Stabbing interval chains

We now recall the problem of stabbing interval chains and the bounds obtained in [3].

Let $[i, j]$ denote the interval of integers $\{i, i + 1, \dots, j\}$. An *interval chain* of size k (also called a *k-chain*) is a sequence of k consecutive, disjoint, nonempty intervals

$$C = I_1 I_2 \cdots I_k = [a_1, a_2][a_2 + 1, a_3] \cdots [a_k + 1, a_{k+1}],$$

where $a_1 \leq a_2 < a_3 < \cdots < a_{k+1}$. We say that a j -tuple of integers (p_1, \dots, p_j) *stabs* an interval chain C if each p_i lies in a different interval of C .

The problem is to stab, with as few j -tuples as possible, all interval chains of size k that lie within a given range $[1, n]$. We let $\mathcal{Z}_k^{(j)}(n)$ denote the minimum size of a collection Z of j -tuples that stab all k -chains that lie in $[1, n]$.

Alon et al. showed in [3] that, for every fixed $j \geq 3$, once k is large enough, $\mathcal{Z}_k^{(j)}(n)$ has near-linear lower and upper bounds roughly of the form $n\alpha_m(n)$, where m grows with k . Specifically:

Theorem C.2 (Interval-chain lower bounds [3]). *Let $j \geq 3$ be fixed, and let $t = \lfloor j/2 \rfloor - 1$. Then there exists a function $Q_j(m)$ of the form*

$$Q_3(m) = 2m + 1, \quad Q_4(m) = \Omega(2^m),$$

and, in general,

$$Q_j(m) \geq \begin{cases} 2^{(1/t!)m^t - O(m^{t-1})}, & j \text{ even}; \\ 2^{(1/t!)m^t \log_2 m - O(m^t)}, & j \text{ odd}; \end{cases}$$

such that, for all $m \geq 3$, if $k \leq Q_j(m)$ then

$$\mathcal{Z}_k^{(j)}(n) \geq c_j n \alpha_m(n) \quad \text{for all } n \geq A(m + c'_j), \tag{7}$$

for some constants c_j and c'_j that depend only on j .

Theorem C.3 (Interval-chain upper bounds [3]). *Let $j \geq 3$ be fixed, and let $t = \lfloor j/2 \rfloor - 1$. Then there exists a function $P_j(m)$ of the form*

$$P_3(m) = 2m, \quad P_4(m) = O(2^m),$$

and, in general,

$$P_j(m) \leq \begin{cases} 2^{(1/t!)m^t + O(m^{t-1})}, & j \text{ even}; \\ 2^{(1/t!)m^t \log_2 m + O(m^t)}, & j \text{ odd}; \end{cases}$$

⁸Actually, $\alpha_{\alpha(x)-3}(x)$ grows to infinity with x , though we omit the proof.

such that, for all $m \geq 3$, if $k \geq P_j(m)$ then

$$\mathcal{Z}_k^{(j)}(n) \leq c_j'' n \alpha_m(n) \quad \text{for all } n, \quad (8)$$

for some constants c_j'' that depend only on j .

Remark: Alon et al. [3] stated the lower bounds of Theorem C.2 in a somewhat weaker form: They just stated that, for every fixed j and m , inequality (7) holds for all $n \geq n_0(j, m)$, without specifying $n_0(j, m)$. The explicit form given above is easily verified by examining the proofs in [3], and we will need this dependence in our arguments below.

C.3 Proof of the lower bounds

Lemma C.4. *Given $r > 1$, let N be a weak $\frac{1}{r}$ -net for $D_s = D_s(n)$, for $n = n(r)$ large enough. Let $\ell = |N|$. Then ℓ must satisfy*

$$\ell \geq \mathcal{Z}_{4d\ell/r}^{(d)}(\ell).$$

Proof. For each point $x \in N$ and each coordinate $1 \leq j \leq d$, mark as “bad” the two points of D_s that surround x when the points are projected into the j -th coordinate. Thus, at most $2d\ell$ points of S are marked “bad”.

Partition D_s into $4d\ell$ contiguous blocks of size $n/(4d\ell)$ each (we can safely ignore the rounding to integers if n is large enough). Then there are $2d\ell$ blocks $B_1, \dots, B_{2d\ell}$ which are “good”, in the sense that they do not contain any bad points. Place $2d\ell - 1$ abstract “separators” $Y_1, \dots, Y_{2d\ell-1}$ between these blocks, such that Y_i lies between B_i and B_{i+1} .

Let $k = 4d\ell/r$. There is a natural one-to-one correspondence between sets \mathcal{B} of k good blocks, and $(k - 1)$ -chains \mathcal{B}' on the separators. Namely, for every $i_1 < i_2 < \dots < i_k$ we map

$$\mathcal{B} = \{B_{i_1}, \dots, B_{i_k}\} \leftrightarrow \mathcal{B}' = [Y_{i_1}, Y_{i_2-1}][Y_{i_2}, Y_{i_3} - 1] \cdots [Y_{i_{k-1}}, Y_{i_k-1}],$$

where the notation $[Y_a, Y_b]$ means $\{Y_a, Y_{a+1}, \dots, Y_b\}$.

Let $\mathcal{B} = \{B_{i_1}, \dots, B_{i_k}\}$ be an arbitrary such set. Let $D'_s = B_{i_1} \cup \dots \cup B_{i_k} \subseteq D_s$. Since $|D'_s| = n/r$ and N is a weak $\frac{1}{r}$ -net for D_s , it follows that $\text{conv } D'_s$ must contain some point $x \in N$. By Carathéodory’s theorem, x is contained in the convex hull of some $d + 1$ points of D'_s ; let these points be q_0, \dots, q_d from left to right.

Recall that for each coordinate $1 \leq j \leq d$, the projection of x into the j -th coordinate falls between two bad points of D_s . Therefore, all the projections of x fall *between* good blocks, and so we can associate with x a d -tuple of separators

$$x' = (Y_{a_1}, \dots, Y_{a_d}).$$

Furthermore, none of the points q_0, \dots, q_d are bad, and therefore they are *far apart* from x in each coordinate. Therefore, Lemmas 1.3 and 1.4 apply, and so the j -th coordinate of x must lie between the j -th coordinates of q_{j-1} and q_j , for every $j = 1, 2, \dots, d$.

It follows that q_0, \dots, q_d belong to $d + 1$ *distinct* blocks B'_0, \dots, B'_d of \mathcal{B} , and furthermore, their relative order with the separators of x' is

$$B'_0, Y_{a_1}, B'_1, Y_{a_2}, \dots, Y_{a_d}, B'_d.$$

In other words, the d -tuple x' stabs the $(k - 1)$ -chain \mathcal{B}' .

Thus, N must have enough points to stab all $(k-1)$ -chains (and so all k -chains) with d -tuples in the range $[1, 2d\ell - 1] \supseteq [1, \ell]$. Therefore,

$$\ell = |N| \geq \mathcal{Z}_k^d(\ell) = \mathcal{Z}_{4d\ell/r}^d(\ell). \quad \square$$

Corollary C.5. *Let $s_d = \max\{c'_d + 1, \lceil 1/c_d \rceil\}$ for the constants c_d and c'_d of Theorem C.2. Then the quantity ℓ of Lemma C.4 must satisfy*

$$\ell \geq \frac{1}{4d} r \cdot Q_d(\alpha(r) - s_d),$$

for the function Q_d of Theorem C.2.

Note that Corollary C.5 implies Theorem 1.6(i).

Proof. Suppose for contradiction that

$$\ell < \frac{1}{4d} r \cdot Q_d(\alpha(r) - s_d). \quad (9)$$

Again let $k = 4d\ell/r$. Let m be the smallest integer such that $k \leq Q_d(m)$. Then $k > Q_d(m-1)$, so by (9),

$$Q_d(m-1) < k < Q_d(\alpha(r) - s_d),$$

and so we must have

$$m \leq \alpha(r) - s_d. \quad (10)$$

Since a weak $\frac{1}{r}$ -net must trivially have at least r points, we have $\ell \geq r$, so

$$\alpha(\ell) \geq \alpha(r) \geq m + s_d \geq m + c'_d + 1,$$

which implies that $\ell > A(m + c'_d)$. Thus, the condition “ $n \geq A(m + c'_j)$ ” of Theorem C.2 is satisfied with our choice of parameters.

Therefore, by Theorem C.2 and Lemma C.4,

$$\ell \geq \mathcal{Z}_k^{(d)}(\ell) \geq c_d \ell \alpha_m(\ell). \quad (11)$$

By (10) we have $m \leq \alpha(r) - \lceil 1/c_d \rceil$, so by (11) and (6),

$$\frac{1}{c_d} \geq \alpha_m(\ell) \geq \alpha_m(r) \geq \alpha_{\alpha(r) - \lceil 1/c_d \rceil}(r) \geq \frac{1}{c_d} + 3,$$

which is a contradiction. □

C.4 Proof of the upper bounds

Lemma C.6. *The set $D_s = D_s(n)$ has a weak $\frac{1}{r}$ -net of size at most $\mathcal{Z}_{\ell/r-1}^{(d)}(\ell)$, where ℓ is a free parameter.*

Proof. Given ℓ , partition D_s into ℓ equal-sized blocks B_1, B_2, \dots, B_ℓ of consecutive points, leaving a pair of points $Y_i = \{y_i, y'_i\}$ between every two adjacent blocks B_i, B_{i+1} . We call the pairs of points $Y_1, \dots, Y_{\ell-1}$ “separators”. We assume ℓ is much smaller than n , so the size of each block B_i can be approximated by n/ℓ .

Consider a set $D'_s \subset D_s$ of size at least n/r . D'_s must contain a set $Q = \{q_1, \dots, q_k\}$ of $k = \ell/r$ points lying on k different blocks $\mathcal{B} = \{B_{i_1}, \dots, B_{i_k}\}$, $i_1 < \dots < i_k$. These blocks define a $(k-1)$ -chain of separators

$$\mathcal{B}' = [Y_{i_1}, Y_{i_2-1}][Y_{i_2}, Y_{i_3-1}] \cdots [Y_{i_{k-1}}, Y_{i_k-1}].$$

Let Z be an optimal family of d -tuples of separators that stab all $(k-1)$ -chains of separators. We have $|Z| = \mathcal{Z}_{k-1}^{(d)}(\ell-1)$.

There must be a d -tuple $z = (Y_{a_1}, \dots, Y_{a_d}) \in Z$ that stabs \mathcal{B}' . Therefore, there exist $d+1$ blocks $B'_0, \dots, B'_d \in \mathcal{B}$ such that the order between them and the elements of z is

$$B'_0, Y_{a_1}, B'_1, Y_{a_2}, \dots, Y_{a_d}, B'_d.$$

Let q'_i be the point of Q that lies in block B'_i , for $0 \leq i \leq d$.

Translate the d -tuple z into a point $z' = (z'_1, \dots, z'_d) \in \mathbb{R}^d$ such that, for each $1 \leq i \leq d$, the coordinate z'_i lies between the i -th coordinates of the two points y_{a_i}, y'_{a_i} that constitute Y_{a_i} .

Then, since z' is far from each of q'_0, \dots, q'_d , it follows from Lemmas 1.3 and 1.4 that $z' \in \text{conv}\{q'_0, \dots, q'_d\} \subseteq \text{conv } D'_s$.

Thus, the set $Z' \subset \mathbb{R}^d$ of all these points z' for every $z \in Z$ is a weak $\frac{1}{r}$ -net for D_s , and it has the desired size. \square

Proof of Theorem 1.6(ii). Take $\ell = r(1 + P_d(\alpha(r)))$, with P_d as in Theorem C.3. Then

$$\mathcal{Z}_{\ell/r-1}^{(d)}(\ell) = \mathcal{Z}_{P_d(\alpha(r))}^{(d)}(\ell) \leq c_d'' \ell \alpha_{\alpha(r)}(\ell),$$

which can be shown to be at most $4c_d'' \ell$ by a simple argument. \square