Discrete Optimization
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Exam solutions

EPFL Lausanne
Spring 2012

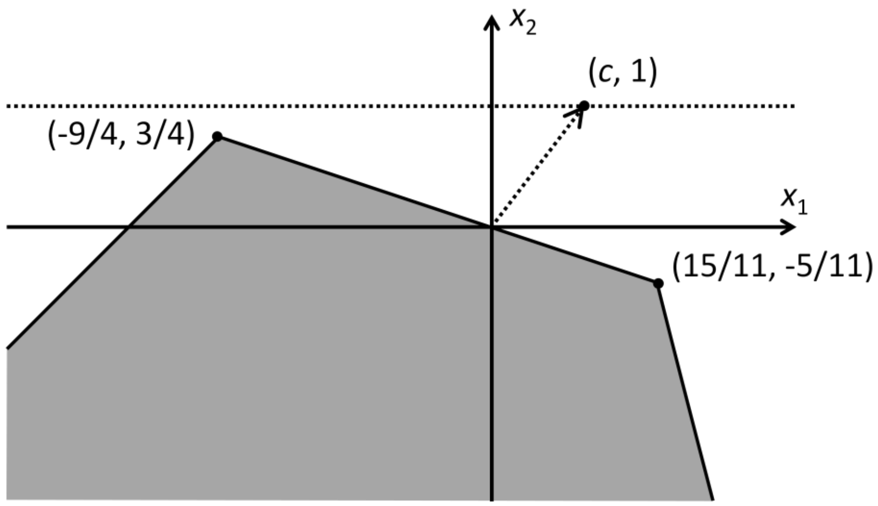
Exercise 1 (9 points)

Consider the following linear program, in which c is an unspecified constant:

$$\begin{aligned} \text{Maximize} \quad & cx_1 + x_2 \\ \text{subject to:} \quad & -x_1 + x_2 \leq 3 \\ & x_1 + 3x_2 \leq 0 \\ & 4x_1 + x_2 \leq 5 \end{aligned}$$

a) For which values of c is this program unbounded?
b) For which values of c is there a unique optimal solution?

Solution:



a) The program is unbounded for $c < -1$ and for $c > 4$.
b) The program has a unique optimal solution when $-1 < c < 1/3$ [in which case the optimum is $(-9/4, 3/4)$] and when $1/3 < c < 4$ [in which case the optimum is $(15/11, -5/11)$].

Exercise 2 (9 points)

Write a linear program in x_1, x_2, x_3 for which the set of feasible solutions is the octahedron with vertices (±1, 0, 0), (0, ±1, 0), (0, 0, ±1) and for which the optimal solution is the edge joining vertices (1, 0, 0) and (0, 1, 0).

Solution:

Maximize $x_1 + x_2$ [or $x_1 + x_2 + cx_3$ for any $-1 < c < 1$]

subject to:

$$
\begin{aligned}
x_1 + x_2 + x_3 &\leq 1\\
x_1 + x_2 - x_3 &\leq 1\\
x_1 - x_2 + x_3 &\leq 1\\
x_1 - x_2 - x_3 &\leq 1\\
-x_1 + x_2 + x_3 &\leq 1\\
-x_1 + x_2 - x_3 &\leq 1\\
-x_1 - x_2 + x_3 &\leq 1\\
-x_1 - x_2 - x_3 &\leq 1
\end{aligned}
$$

[The octahedron has 8 triangular facets. Each facet contains one vertex among (±1, 0, 0), one vertex among (0, ±1, 0), and one vertex among (0, 0, ±1). For each facet we need a linear constraint that is satisfied with equality by the facet's vertices and is satisfied by all the polytope's vertices.]

Exercise 3 (10 points)

Express the following two problems as linear programs (you do not need to solve the linear programs):

a) A hospital needs to have at least the following number of nurses on duty each day:

0:00 – 4:00	4:00 – 8:00	8:00 – 12:00	12:00 – 16:00	16:00 – 20:00	20:00 – 0:00
5 nurses	8 nurses	10 nurses	20 nurses	15 nurses	10 nurses

Each nurse reports daily at either 0:00, 4:00, 8:00, 12:00, 16:00, or 20:00 and works for 8 consecutive hours. The same schedule repeats day after day. We would like to find the schedule that minimizes the total number of nurses.

Solution:

Let x_i denote the number of nurses that report at time i.

Minimize $x_0 + x_4 + x_8 + x_{12} + x_{16} + x_{20}$

subject to:

$x_0 + x_4 \geq 8$
$x_4 + x_8 \geq 10$
$x_8 + x_{12} \geq 20$
$x_{12} + x_{16} \geq 15$
$x_{16} + x_{20} \geq 10$
$x_{20} + x_0 \geq 5$

$x_0, x_4, x_8, x_{12}, x_{16}, x_{20} \geq 0$

b) A paper mill produces paper rolls with a standard width of 3 meters. It receives the following order:

- 1700 rolls of width 131 cm;
- 2500 rolls of width 85 cm;
- 1400 rolls of width 80 cm;
- 3100 rolls of width 62 cm.

We would like to find the smallest number of 3m rolls that have to be cut to satisfy this order, and how to cut them.

Solution:

There are 16 ways of cutting a roll:

1. (131, 131)
2. (131, 85, 80)
3. (131, 85, 62)
4. (131, 80, 80)
5. (131, 80, 62)
6. (131, 62, 62)
7. (85, 85, 85)
8. (85, 85, 80)
9. (85, 85, 62, 62)
10. (85, 80, 80)
11. (85, 80, 62, 62)
12. (85, 62, 62, 62)
13. (80, 80, 80)
14. (80, 80, 62, 62)
15. (80, 62, 62, 62)
16. (62, 62, 62, 62)

(We only consider ways in which the leftover width is less than 62cm.)

Let x_i be the number of rolls cut in way *i*. Then the linear program is:

Minimize $x_1 + x_2 + \ldots + x_{16}$

subject to:

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 1700$$
$$x_2 + x_3 + 3x_7 + 2x_8 + 2x_9 + x_{10} + x_{11} + x_{12} \geq 2500$$
$$x_2 + 2x_4 + x_5 + x_8 + 2x_{10} + x_{11} + 3x_{13} + 2x_{14} + x_{15} \geq 1400$$
$$x_3 + x_5 + 2x_6 + 2x_9 + 2x_{11} + 3x_{12} + 2x_{14} + 3x_{15} + 4x_{16} \geq 3100$$

$$x_1, x_2, \ldots, x_{16} \geq 0$$

Exercise 4 (9 points)

We would like to solve the following linear program with the simplex method:

Minimize $x_1 - 3x_2 + 2x_3$

subject to:
$2x_1 - x_3 = 7$
$3x_1 + 4x_2 \geq 5$
$x_1 + x_2 + 2x_3 \leq 30$

$x_1, x_2 \geq 0$

a) Express the linear program in equational form.

b) Which ***auxiliary linear program*** do we have to solve in order to find an initial feasible basis for our linear program?

Solution:

a) We negate the objective function to turn the "minimize" into "maximize", we replace x_3 by $x_4 - x_5$, and we add slack variables x_6 and x_7 to the second and third constraints, respectively:

Maximize $-x_1 + 3x_2 - 2x_4 + 2x_5$

subject to:
$2x_1 - x_4 + x_5 = 7$
$3x_1 + 4x_2 - x_6 = 5$
$x_1 + x_2 + 2x_4 - 2x_5 + x_7 = 30$

$x_1, x_2, x_4, x_5, x_6, x_7 \geq 0$

b) We add a slack variable to each equality constraint, and we minimize their sum:

Maximize $-x_8 - x_9 - x_{10}$

subject to:
$2x_1 - x_4 + x_5 + x_8 = 7$
$3x_1 + 4x_2 - x_6 + x_9 = 5$
$x_1 + x_2 + 2x_4 - 2x_5 + x_7 + x_{10} = 30$

$x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \geq 0$

Exercise 5 (9 points)

Consider the following simplex tableau:

$$
\begin{array}{lcl}
x_2 & = & 7 + 3x_1 - 5x_3 - 3x_4 \\
x_5 & = & 2x_1 - x_3 + 2x_4 \\
x_6 & = & -2x_1 + 2x_3 \\
\hline
z & = & 10 - x_1 + x_3 + x_4
\end{array}
$$

a) What is the feasible solution corresponding to this tableau, and what is the corresponding value of the objective function?

b) What are the possible pivot steps from this tableau? Which of these pivot steps are degenerate?

c) Perform one of the pivot steps from part b) and write down the new tableau.

Solution:

a) (0, 7, 0, 0, 0, 0); 10.

b) x_3 enters the base and x_5 leaves (degenerate); or x_4 enters and x_2 leaves (nondegenerate).

c)

$$
\begin{array}{lcl}
x_2 & = & 7 - 7x_1 - 13x_4 + 5x_5 \\
x_3 & = & 2x_1 + 2x_4 - x_5 \\
x_6 & = & 2x_1 + 4x_4 - 2x_5 \\
\hline
z & = & 10 + x_1 + 3x_4 - x_5
\end{array}
$$

or:

$$
\begin{array}{lcl}
x_4 & = & 7/3 + x_1 - 1/3x_2 - 5/3x_3 \\
x_5 & = & 14/3 + 4x_1 - 2/3x_2 - 13/3x_3 \\
x_6 & = & -2x_1 + 2x_3 \\
\hline
z & = & 37/3 - 1/3x_2 - 2/3x_3
\end{array}
$$

Exercise 6 (10 points)

We claimed in class that the problem of finding an optimal solution of a linear program is ***not harder*** than the problem of finding an arbitrary feasible solution. Meaning: If we were to have an oracle that, given an LP, returns a feasible solution, we could use this oracle to find optimal solutions to linear programs.
We gave in class ***two different arguments*** in support of this claim. What are they? (One argument gives 7 points.)

Solution:

First argument: Perform a binary search on the objective function. Suppose, for example, that we are given a maximization problem, and that we know in advance (from other considerations) that the objective function is nonnegative and cannot be more than 100. Then we add to the linear program a constraint saying that the objective function must be at least 50 (this is a linear constraint so it is allowed). We query the oracle with this LP. If the oracle returns a feasible solution, then we try requiring that the objective function be at least 75; if not, then we try with 25; and so on. We perform a binary search in this way for enough steps, until we have a solution that is very close to optimal.

Second argument: Using duality. Suppose the given LP is a maximization problem. Then we put together the given LP, its dual LP, and an additional constraint requiring that the primal objective function be at least the dual objective function. We ask the oracle for a feasible solution to this program.

Specifically, let the given LP be "maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$". Then we ask the oracle for a feasible solution to the following set of constraints (where $\mathbf{x}$ and $\mathbf{y}$ are the unknowns):

$A\mathbf{x} \le \mathbf{b}$,
$A^T\mathbf{y} \ge \mathbf{c}$,
$\mathbf{c}^T\mathbf{x} \ge \mathbf{b}^T\mathbf{y}$,
$\mathbf{x}, \mathbf{y} \ge \mathbf{0}$.

Every feasible solution to this system is an optimal solution to the given LP.

Exercise 7 (11 points)

a) Complete the following version of the Farkas lemma:

For every *mxn* matrix A and every vector $\mathbf{b}$ in $\mathbf{R}^m$, ***either*** the system $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution $\mathbf{x} \geq \mathbf{0}$ ***or*** …

b) Let $p_1 = (a_1, b_1, c_1)$, $p_2 = (a_2, b_2, c_2)$, …, $p_5 = (a_5, b_5, c_5)$ be five points in $\mathbf{R}^3$, and let $q = (a, b, c)$ be another point in $\mathbf{R}^3$. Write a system of the form $A\mathbf{x} = \mathbf{b}$ that has a nonnegative solution if and only if q lies in the convex hull of $p_1, \ldots, p_5$.

c) Use the Farkas lemma in part a) to prove that either q lies in the convex hull of $p_1, \ldots, p_5$ or there exists a plane in $\mathbf{R}^3$ that separates q from $p_1, \ldots, p_5$.

Solution:

a) ***… or*** there exists a "simple proof" that such a system has no solution; namely, there exists a vector $\mathbf{y}$ in $\mathbf{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b} < 0$.

b) Point q belongs to the convex hull of $p_1, \ldots, p_5$ if and only if it is a ***convex combination*** of them; namely, if and only if there exist nonnegative real numbers $x_1, \ldots, x_5$ such that $x_1p_1 + \ldots + x_5p_5 = q$ and $x_1 + \ldots + x_5 = 1$.
Therefore, the desired system is:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}.$$

c) If the above system has no nonnegative solution, then by part a) there exists a vector $\mathbf{y}$ such that, on the one hand,

$$[y_1 \quad y_2 \quad y_3 \quad y_4] \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \geq [0 \quad 0 \quad 0 \quad 0 \quad 0],$$

while, on the other, $y_1a + y_2b + y_3c + y_4 < 0$.
This means that points $p_1, \ldots, p_5$ are on one side of the plane

$$\{(A, B, C) : Ay_1 + By_2 + Cy_3 = -y_4\},$$

and q is on the other side of this plane.

Exercise 8 (11 points)

We are given a bipartite graph with vertex sets V_1, V_2, $|V_1| = |V_2| = n$, and edge set $E \subseteq V_1 \times V_2$. Each edge $e \in E$ is assigned a nonnegative weight w_e. We are interested in finding a maximum-weight perfect matching. (Recall: A perfect matching is a set of n edges such that each vertex from V_1 and each vertex from V_2 is incident to exactly one edge.)

a) Express this problem as an integer program.

b) Appropriately relax the integer program in part a) into a linear program.

c) Prove that the linear program has an optimal solution that satisfies the original integer program.

Solution:

a) For each edge e, introduce a variable x_e, which equals 1 if e is part of the perfect matching, and 0 otherwise. Then the desired integer program is:

Maximize $\sum w_e x_e$

subject to: $\sum_{e\,:\,e \text{ incident to } v} x_e = 1$ for each v in V_1, V_2,

$x_e = 0$ or 1 for each e.

b) The relaxed linear program is:

Maximize $\sum w_e x_e$

subject to: $\sum_{e\,:\,e \text{ incident to } v} x_e = 1$ for each v in V_1, V_2,

$0 \le x_e \le 1$ for each e.

c) Let $\mathbf{x}^*$ be an optimal solution for the linear program that has the minimum possible number of noninteger components. Suppose for a contradiction that $\mathbf{x}^*$ has at least one noninteger component.
Let e be an edge for which x^*_e is noninteger. Let v be one of the endpoints of e. Then v must have another incident edge e' for which $x^*_{e'}$ is also noninteger. Consider the other endpoint v' of e'. It must have another incident edge e'' for which $x^*_{e''}$ is also noninteger. Consider its other endpoint v'', and so on. We continue in this fashion, until we return to a previously visited vertex, and we obtain a cycle of noninteger edges. This cycle must have even length, since our graph is bipartite.

Alternatingly add +ε and -ε to the variables x_e of the edges around the cycle. The vertex constraints of the LP are still satisfied. Continuously increase ε, starting from 0, until the first x_e in the cycle becomes either 0 or 1. We obtain a new feasible solution for the LP. The objective function could not have increased, because we assumed that **x*** is optimal. The objective function could not have decreased either, because then we could have *decreased* ε starting from 0, and then the objective function would have increased. Therefore, the objective function stayed the same. We have thus obtained another optimal solution, which has *fewer* noninteger components than **x***; contradiction.

Exercise 9 (8 points)

Recall that an *n-bit code with distance d* is a subset $C \subseteq \{0,1\}^n$ such that every two words in *C* differ in at least *d* bits. Suppose *d* = 2*r*+1 is an odd number. Prove that no *n*-bit code with distance *d* can have more than

$$\frac{2^n}{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r}}$$

words.

Solution:

For each word *w* in $\{0, 1\}^n$ let *B*(*w*), the *ball around w of radius r*, be the set of words *w'* that differ from *w* in at most *r* bits. Clearly, $|B(w)| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r}$.

We claim that for every two words *w* and *w'* in *C*, the corresponding balls *B*(*w*) and *B*(*w'*) must be disjoint. Indeed, suppose for a contradiction that some word *w''* belongs to both *B*(*w*) and *B*(*w'*). Then *w* differs from *w''* in at most *r* bits, and *w''* differs from *w'* in at most *r* bits, so *w* and *w'* differ in at most 2*r* bits; contradiction.

Therefore, the balls are all disjoint, and the claim follows.

Exercise 10 (7 points)

Write the dual of the following linear program:

Maximize $-2x_1 + 3x_2 - 7x_4$

subject to:
$x_1 - x_2 + 2x_3 = 5$
$3x_1 + 4x_4 \leq 7$
$2x_2 - x_3 - x_4 \geq 6$

$x_1, x_2 \geq 0$
$x_4 \leq 0$

Solution:

Minimize $5y_1 + 7y_2 + 6y_3$

subject to:
$y_1 + 3y_2 \geq -2$
$-y_1 + 2y_3 \geq 3$
$2y_1 - y_3 = 0$
$4y_2 - y_3 \leq -7$

$y_2 \geq 0$
$y_3 \leq 0$

Exercise 11 (7 points)

Consider the following system:

$2x$			$-$	z	$\leq$	9		(a)
$-x$	$+$	$2y$	$-$	z	$\leq$	5		(b)
x	$-$	$3y$			$\leq$	-7		(c)
x	$-$	$4y$			$\leq$	-6		(d)
		y	$+$	$3z$	$\leq$	0		(e)
$4x$	$+$	y	$+$	$2z$	$\leq$	-3		(f)

Perform Fourier–Motzkin elimination on the variable z. Indicate how the equations in the new system can be obtained as linear combinations of the equations in the old system.

Solution:

Equations (a) and (b) have z with a negative coefficient, equations (c) and (d) do not have z, and equations (e) and (f) have z with a positive coefficient. We combine each equation in {(a), (b)} with each equation in {(e), (f)} so as to cancel out z. We leave equations (c) and (d) unchanged:

$6x$	$+$	y	$\leq$	27	3(a) + (e)
$8x$	$+$	y	$\leq$	15	2(a) + (f)
$-3x$	$+$	$7y$	$\leq$	15	3(b) + (e)
$2x$	$+$	$5y$	$\leq$	7	2(b) + (f)
x	$-$	$3y$	$\leq$	-7	(c)
x	$-$	$4y$	$\leq$	-6	(d)