Duality in Linear Programming

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Idea

Idea: transform an INITIAL problem (P) to a DUAL problem which is hoped to have the following properties:

1. It's easier to solve
2. Its solutions give informations (or why not, solve) the initial problem

Starting point

We have objective function to maximize, and constraints. We are looking for naturally an upper bound easy to obtain when considering the constraints. So we are playing with constraints, making allowed linear combinaisons of them to obtain this bound.

Example 1:(silly)

$$Max\ 2x_1 + x_2 \quad |$$

$$4x_1 + 2x_2 = 2$$
$$x_i \geq 0$$

A trivial upper bound to the problem is 1.

Examples

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$$Max\ 2x_1 + x_2 \ \ |$$

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A trivial upper bound to the problem is 1.

Example 2:(less silly)

$$Max\ 2x_1 + 3x_2 \ \ |$$

$$x_1 + x_2 \leq 1$$
$$x_2 \leq 1$$
$$x_i \geq 0$$

We see that $2(1) + 1(2) \Rightarrow$ the max is less than 3 (and is effectively 3, by taking $x_1 = 0, x_2 = 1$.

$$Max\ 2x_1 + 3x_2 \quad |$$

$$\begin{array}{rcll} x_1 + x_2 & \leq & 1 & (C_1) \\ x_2 & \leq & 1 & (C_2) \\ x_i & \geq & 0 & (C_3) \end{array}$$

Take y_1, y_2 positive (to keep the inequalities in the same sense) so we have:

$$y_1 \star (C_1) + y_2 \star (C_2) \leq f(y_1, y_2)(= 1y_1 + 1y_2\ here)$$

Moreover,

$$y_1(C_1) + y_2(C_2) = x_1(y_1 + 0y_2) + x_2(y_1 + y_2)$$

so if $2 \leq (y_1 + 0y_2)$ and $3 \leq (y_1 + y_2)$ then
$2x_1 + 3x_2 \underbrace{\leq}_{positivity\ of\ x_i} x_1(y_1 + 0y_2) + x_2(y_1 + y_2) \leq f(y_1, y_2)$.

So under the constraints

$$y_1 + 0y_2 \geq 2 \tag{1}$$
$$y_1 + y_2 \geq 3 \tag{2}$$

the function $f(y_1, y_2)$ gives an upper bound for my initial problem.

Examples

As we want the smallest upper bound, what we are looking for is

$$Min\ f(y_1, y_2)\ \ |$$

$$\begin{aligned} y_1 + 0y_2 &\geq 2 \\ y_1 + y_2 &\geq 3 \\ y_i &\geq 0 \end{aligned}$$

Examples

Writing the initial problem

$$(P) : Max\ c^T x | Ax \leq b, x \geq 0$$

we see that the dual problem is

$$(D) : Min\ b^T y | A^T y \geq c, y \geq 0$$

Duality for an equational form LP

$$Initial(P) : Max\ c^T x | Ax = b, x \geq 0$$

$$Dual(D) : \quad Min\ b^T y | A^T y \geq c, y \in \mathbb{R}$$

Duality for an equational form LP

Indeed, if we note $a_1^T ... a_m^T$ the rows of A , and $b = \begin{pmatrix} b_1 \\ ... \\ b_m \end{pmatrix}$ the constraints are

$$a_1^T x = b_1, ..., a_m^T x = b_m$$

so if we multiply by $y_1 ... y_m$ which are real arbitrary since an equality stays an equality whatever the values you're multiplying by, we find

$$\underbrace{(y_1 a_1^T + ... + y_m a_m^T)x}_{c'^T x} = \underbrace{y_1 b_1 + ... + y_m b_m}_{z = b^T y}$$

If $c' \geq c$ (ie. on each component) by positivity of x we have

$$c^T x \leq c'^T x = z$$

so to obtain the best z possible, we have to minimize $z = z(y_1, ..., y_m)$ under constraint $c' \geq c$ this means

$$Min\ z = b^T y \mid c' = A^T y \geq c.$$

Weak duality

weak duality theorem

For each feasible solution y of (D) the value $b^T y$ provides an upper bound on the maximum of the objective function for (P). So, for each solution x of (P) and each feasible solution y of (D) we have

$$c^T x \leq b^T y.$$

In particular

1. (P) unbounded $\Rightarrow$ (D) is infeasible
2. (D) unbounded $\Rightarrow$ (P) is infeasible

It follows directly from what we saw before!

Strong duality

Strong duality theorem

$$Initial(P): Max\ c^T x | Ax = b, x \geq 0$$

$$Dual(D): \quad Min\ b^T y | A^T y \geq c, y \in \mathbb{R}$$

Then one and only one of the following situation occurs:

- Neither (P) nor (D) has feasible solution
- (P) unbounded and (D) has no feasible solution
- (D) unbounded and (P) has no feasible solution
- Both (P) and (D) have a feasible solution. Then both have an optimal solution, and if x^* is an optimal solution for (P) and y^* is an optimal solution for (D) then

$$c^T x^* = b^T y^*.$$

Cf. Gabriel's notes, Matousek& Gartner 's book, chvatal's book.

Dualization in the non equational form

Constraints now are $C_1 ... C_m$

$$a_{i1}x_1 + ... + a_{in}x_n \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} b_i \quad (C_i).$$

The dual problem is associating one variable y_i to each C_i and the following chart is summarizing the situation

Dualization in the non functionnal form

.	(P)	(D)
Variables	$x_1 \ldots x_n$	$y_1 \ldots y_m$
Matrice	A	A^T
Right Hand Side	b	c
Objective	Max $c^T x$	Min $b^T y$
Constraints	i^e constraint is $\leq$	y_i is ≥ 0
	i^e constraint is $\geq$	y_i is ≤ 0
	i^e constraint is $=$	y_i is $\in \mathbb{R}$
	x_j is ≥ 0	j^e constraint is $\geq$
	x_j is ≤ 0	j^e constraint is $\leq$
	x_j is $\in \mathbb{R}$	j^e constraint is $=$

Feasible vs Optimal

Remark

Finding an optimal solution is no more complicated than finding a feasible one.

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Essentially, this is because thanks to strong duality theorem, solving (P) is equivalent to solving

$$(\tilde{P}): \quad Max\ c^T x \quad \Big|$$

$$\begin{aligned} Ax &= b \\ x &\geq 0 \\ A^T y &\geq c \\ y &\in \mathbb{R} \\ c^T x &\geq b^T y \end{aligned}$$

And a solution $(\tilde{x}, \tilde{y})$ of $(\tilde{P})$ gives you an optimal solution $\tilde{x}$ of (P).

To go further

The most important theorem in (convex) optimisation is Karush-Kuhn-Tucker's one. The duality is present everywhere in this theory.

One important thing in general duality (not only linear programming) is the fact that the dual objective is convex.