Discrete Optimization – Lecture Notes handout 2

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In these lecture notes we prove the *max-flow min-cut theorem* using the LP duality theorem.

Let $G = (V, E)$ be a directed graph with a node $s \in V$, called the *source*, that has only outgoing edges, and a note $t \in V$, called the *sink*, that has only incoming edges.

Assume that each edge e has a nonnegative capacity assigned to it.

A *flow* $\mathcal{F}$ in G is an assignment of nonnegative numbers, called *flows*, to the edges such that

- the flow along each edge is at most the capacity of that edge,
- the net flow in each vertex besides s or t is zero.

The *value* of $\mathcal{F}$ is the sum of the flows of the edges entering t, or equivalently, the sum of the flows of the edges leaving s.

A *cut* in G is a partition of V into two sets S and T such that $s \in S$ and $t \in T$. The *capacity* of the cut is the sum of the capacities of the edges that go from S to T. See Figure 1.

Given a flow $\mathcal{F}$ in G and a cut (S, T) in G, the value of $\mathcal{F}$ (the total flow entering t) is clearly equal to the total flow going from S to T. It follows that the value of $\mathcal{F}$ must be at most the capacity of the cut (S, T). Therefore:

Lemma 1. *The maximum flow of G is at most the capacity of the minimum cut of G, or:*

$$\textit{max-flow}(G) \le \textit{min-cut}(G).$$

We will now prove that these two quantities are actually equal:

Theorem 2 (Max-flow min-cut theorem)**.** *The maximum flow of G equals the capacity of the minimum cut of G, or:*

$$\textit{max-flow}(G) = \textit{min-cut}(G).$$

We will prove this theorem using LP duality.

Proof. Let G be the given directed graph. The following LP formulates the problem of finding a maximum flow in G:

Maximize

$$\sum_{e\colon e \text{ entering } t} f_e$$

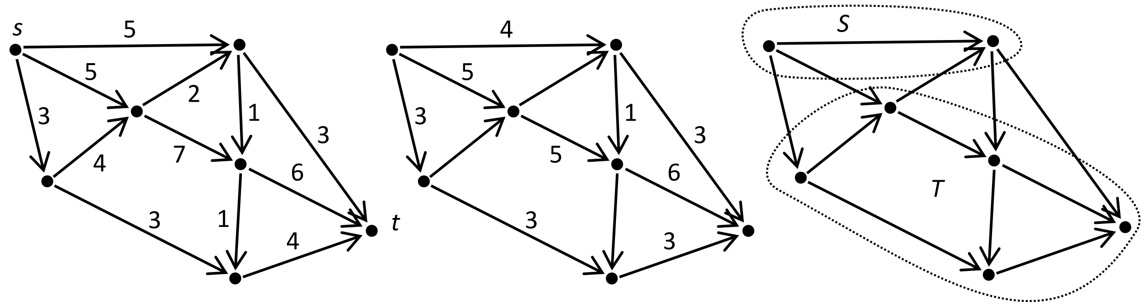

Figure 1: Left: A directed graph with edge capacities. Center: A maximum flow. Right: A minimum cut.

subject to:

$$f_e \le c_e \qquad \text{for every } e \in E, \tag{1}$$

$$\sum_{e\colon e \text{ leaving } v} f_e - \sum_{e\colon e \text{ entering } v} f_e = 0 \qquad \text{for every } v \in V \setminus \{s,t\}. \tag{2}$$

$$f_e \ge 0 \qquad \text{for every } e \in E. \tag{3}$$

The variables of this LP are f_e for all the edges $e \in E$.

Let us formulate the dual LP. We multiply each constraint in (1) by a nonnegative variable $\delta_e \ge 0$, and we multiply each constraint in (2) by an arbitrary variable h_v. When we add up these constraints and group the terms by f_e, we get that for each edge $e = uv$ the corresponding variable f_{uv} is multiplied by $(\delta_{uv} + h_u - h_v)$, except that if $u = s$ then the term h_u is missing, and if $v = t$ then the term h_v is missing. In other words, we obtain the following inequality:

$$\sum_{sv \in E} (\delta_{sv} - h_v) f_{sv} + \sum_{ut \in E} (\delta_{ut} + h_u) f_{ut} + \sum_{uv \in E, u \ne s, v \ne t} (\delta_{uv} + h_u - h_v) f_{uv} \le \sum_{uv \in E} \delta_{uv} c_{uv}. \tag{4}$$

Since we want (4) to be an upper bound for the objective function of the primal LP, we want the coefficients multiplying the variables f_{ut} to be at least 1, and the coefficients multiplying all the other variables f_{uv} to be at least 0. In other words, we need the following constraints:

$$\begin{aligned} \delta_{sv} - h_v &\ge 0 && \text{for every } sv \in E, \\ \delta_{uv} + h_u - h_v &\ge 0 && \text{for every } uv \in E, u \ne s, v \ne t, \\ \delta_{ut} + h_u &\ge 1 && \text{for every } ut \in E. \end{aligned}$$

We can write these constraints more concisely by defining $h_s = 0$, $h_t = 1$. Then the constraints become:

$$\delta_{uv} \ge h_v - h_u \qquad \text{for every } uv \in E.$$

The quantity that we want to minimize in the dual LP is the right-hand side of (4):

$$\text{Minimize} \sum_{uv \in E} \delta_{uv} c_{uv}. \tag{5}$$

Recall that the variables of the dual LP are the δ's, which must be nonnegative, and the h's, which can be arbitrary.

Let us understand the dual LP. Each vertex of $v \in V$ has a "height" h_v assigned to it. The height of s is fixed to 0, the height of t is fixed to 1, and the height of every other vertex is a variable we can play with.

In addition, each edge $uv \in E$ has a variable δ_{uv} which must be at least the difference in height between u and v. Since the dual objective function (5) depends nonnegatively on each δ_{uv}, we can without loss of generality set δ_{uv} as small as possible; meaning, if $h_v \geq h_u$ then we set $\delta_{uv} = h_v - h_u$; otherwise we set $\delta_{uv} = 0$. Thus, in the dual LP we only have to play with the heights h_v. After we fix the heights, the δ's and the objective function are determined.

Clearly, it does not pay to make any height less than 0 or more than 1, because then, increasing to 0 all heights that are less than 0 and decreasing to 1 all heights that are more than 1 only decreases the objective function.

Moreover, we can assume that every height is either 0 or 1. Indeed, suppose that some heights are nonintegers. If increase them or decrease them in tandem, then the objective function changes linearly in response. We move the noninteger heights in tandem in the direction that makes the objective function decrease (or stay the same), until one height becomes 0 or 1. In this way we decreased the number of noninteger heights; we can repeat this until all heights are integers.

Thus, the dual LP corresponds to the following problem: Partition V into two sets of vertices S and T, where S will consist of all vertices with height 0 and T will consist of all vertices of height 1, such that $s \in S$ and $t \in T$. Then $\delta_{uv} = 1$ exactly for those edges with $u \in S$ and $v \in T$, and $\delta_{uv} = 0$ for all the other edges. Therefore the dual objective function (5) equals exactly the capacity of the cut (S, T).

In summary, the primary LP corresponds to maximizing the flow of G and the dual LP corresponds to minimizing a cut in G. Therefore, LP duality implies that max-flow(G) = min-cut(G). □