Discrete Optimization – Lecture Notes handout 1

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In these lecture notes we prove the duality theorem using the simplex algorithm. We do it in a slightly different way than in the textbook, and that is why we are distributing these lecture notes.

Let P be a linear program in equational form:

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

Suppose P is feasible and bounded, and let $\mathbf{x}^*$ be an optimal solution for P.

The linear program dual to P is the following, which we call D:

$$\text{Minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c}.$$

Let $\mathbf{y}^*$ be an optimal solution for D.

The duality theorem states that $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.

In other words, the duality theorem asserts that there exists a way derive a "proof" that $\mathbf{x}^*$ is indeed an optimal solution of P, by taking a linear combination of the constraints of P.

Let us elaborate. The equality constraints of P are

$$\begin{aligned} \mathbf{a}_1^T\mathbf{x} &= b_1 \\ \mathbf{a}_2^T\mathbf{x} &= b_2 \\ &\vdots \\ \mathbf{a}_m^T\mathbf{x} &= b_m \end{aligned}$$

where $\mathbf{a}_1^T, \cdots, \mathbf{a}_m^T$ are the rows of A. A linear combination of these constraints is obtained by multiplying these equations by suitable numbers $y_1, \cdots, y_m$ (which could be positive, negative, or zero) and adding them up, obtaining an equation of the form

$$(c')^T\mathbf{x} = z,$$

where $(c')^T = (y_1\mathbf{a}_1^T + \cdots + y_m\mathbf{a}_m^T)$ and $z = y_1b_1 + ... + y_mb_m$.

Now, if $c' \geq c$ in each coordinate, then this means that the objective function of P cannot be larger than z, because for every feasible solution x we have (since $x \geq 0$)

$$\mathbf{c}^T\mathbf{x} \leq (c')^T\mathbf{x} = z.$$

Thus, in order to obtain a good upper bound on the objective function of P, we want to obtain a z that is as small as possible, subject to the constraint that $c' \geq c$ in each coordinate. And that is exactly what the dual linear program D is doing:

$$\text{Minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c}.$$

A careful look at the simplex algorithm

The simplex algorithm essentially performs a sequence of linear combinations on the equality constraints of P.

If we keep track of the linear operations that the simplex algorithm performs, at the end we can obtain not only an optimal solution $\mathbf{x}^*$ (assuming the LP is feasible and bounded) but also a vector $\mathbf{y}^*$ that proves that $\mathbf{x}^*$ is indeed optimal. In other words, by keeping track of some additional information we can make the simplex algorithm solve the primal and dual linear programs simultaneously.

The best way to explain this is to illustrate it with an example.

Let P be the following linear program in equational form:

Maximize

$$-x_1 + x_2 + 2x_4 \qquad (z_0)$$

subject to

$$\begin{aligned} 3x_1 - 2x_2 + 5x_3 + 4x_4 &= 1 \qquad (E_1) \\ 7x_1 \qquad\;\; + x_3 \;\; + 8x_4 &= 3 \qquad (E_2) \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

where we have labeled the objective function z_0 and the two equations E_1 and E_2.

At some point during the execution of the simplex algorithm, the current tableau could be the following:

$$\begin{array}{rcl} x_1 = \frac{3}{7} \qquad - \frac{1}{7}x_3 - \frac{8}{7}x_4 & & \left(\frac{1}{7}E_2\right) \\ x_2 = \frac{1}{7} \quad + \frac{16}{7}x_3 + \frac{2}{7}x_4 & & \left(-\frac{1}{2}E_1 + \frac{3}{14}E_2\right) \\ \hline z = -\frac{2}{7} + \frac{17}{7}x_3 + \frac{24}{7}x_4 & & \left(z_0 + \frac{1}{2}E_1 - \frac{1}{14}E_2\right) \end{array}$$

In this tableau we have labeled each equation with the linear combination of E_1 and E_2 that gives rise to the equation. For example, the second equation is labeled with "$-\frac{1}{2}E_1 + \frac{3}{14}E_2$". That linear combination gives

$$x_2 - \frac{16}{7}x_3 - \frac{2}{7}x_4 = \frac{1}{7},$$

which is equivalent to the second equation in the tableau after rearranging the terms appropriately.

Furthermore, the objective function is labeled with "$z_0 +$ " followed by a linear combination. That means the following: If we take this linear combination (namely, $\frac{1}{2}E_1 - \frac{1}{14}E_2$) and move all the terms to the left-hand-side, we obtain

$$-\frac{2}{7} + x_1 - x_2 + \frac{17}{7}x_3 + \frac{10}{7}x_4 = 0.$$

If we add the left-hand-side to z_0 (which makes sense since we are actually adding zero) we get

$$-\frac{2}{7} + \frac{17}{7}x_3 + \frac{24}{7}x_4,$$

which is exactly the formula for z in the tableau.

Now let us see how to update the additional information during a pivot step. Let us choose x_4 as the entering variable and x_1 as the leaving variable. Then, in the first equation of the tableau we have to solve for x_4. To do this, we have to multiply the equation by 7/8 and rearrange terms:

$$x_4 = \frac{3}{8} - \frac{7}{8}x_1 - \frac{1}{8}x_3.$$

Therefore, this new equation will be labeled $\frac{7}{8} \cdot \frac{1}{7}E_2 = \frac{1}{8}E_2$.

Now we substitute for x_4 in the second equation. What we are doing actually is to add to the second equation 2/7 times the new equation for x_4. Therefore, the second equation will be labeled

$$-\frac{1}{2}E_1 + \frac{3}{14}E_2 + \frac{2}{7} \cdot \frac{1}{8}E_2 = -\frac{1}{2}E_1 + \frac{7}{28}E_2.$$

Finally, we substitute for x_4 in the objective function. This is equivalent to adding $-24/7$ times the new equation for x_4, where in the equation we put everything on the left-hand-side. Thus, the new objective function has the label

$$z_0 + \frac{1}{2}E_1 - \frac{1}{14}E_2 - \frac{24}{7} \cdot \frac{1}{8}E_2 = z_0 + \frac{1}{2}E_1 - \frac{1}{2}E_2.$$

Thus, the new tableau looks like this:

$$\begin{array}{ll}
x_4 = \frac{3}{8} - \frac{7}{8}x_1 - \frac{1}{8}x_3 & (\frac{1}{8}E_2) \\
x_2 = \frac{1}{4} - \frac{1}{4}x_1 + \frac{9}{4}x_3 & (-\frac{1}{2}E_1 + \frac{7}{28}E_2) \\
\hline
z = 1 \ - 3x_1 + 2x_3 & (z_0 + \frac{1}{2}E_1 - \frac{1}{2}E_2)
\end{array}$$

The algorithm continues applying pivot steps until it reaches the final tableau—the one in which all the variables in the formula for z have nonpositive coefficients. In our case, the final tablau is the following:

$$\begin{array}{ll}
x_2 = 7 - 16x_1 - 18x_4 & (-\frac{1}{2}E_1 + \frac{15}{6}E_2) \\
x_3 = 3 \ - 7x_1 - 8x_4 & (E_2) \\
\hline
z = 7 - 17x_1 - 16x_4 & (z_0 + \frac{1}{2}E_1 - \frac{5}{2}E_2)
\end{array}$$

This tableau corresponds to the optimal solution $\mathbf{x}^* = (0, 7, 3, 0)$, which achieves objective function $\mathbf{c}^T\mathbf{x}^* = 7$.

Now, let $\mathbf{y}^* = (-1/2, 5/2)$ consist of the *negatives* of the coefficients of E_1 and E_2 in the label for the "z" line. We claim that $\mathbf{y}^*$ is the desired optimal solution for the dual linear program.

Let us see why this is so. The expression $-17x_1 - 16x_4$ in the formula for z has all coefficients nonpositive. On the other hand, we know that this expression is obtained by mutliplying E_1 by $1/2$, E_2 by $-5/2$, and adding them to $\mathbf{c}^T\mathbf{x}$. Therefore, the vector $\mathbf{y}^* = (-1/2, 5/2)$ satisfies $A^T\mathbf{y}^* \geq \mathbf{c}$, which is exactly the requirement for $\mathbf{y}$ in the dual linear program.

Similarly, the constant term 7 in the formula for z implies that $\mathbf{b}^T\mathbf{y}^* = 7$, which is the same as $\mathbf{c}^T\mathbf{x}^*$, as desired.