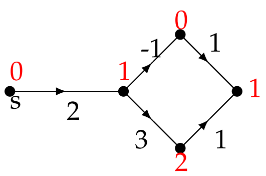
DISCRETE OPTIMIZATION WEEK 10

EXERCISE 1

We are given a directed graph $G = (V, E)$ in which each edge e is assigned a "length" b_e. The lengths may be negative, but we assume that G does not contain a negative cycle (a directed cycle whose total length is negative). We are also given two vertices s, t in V. We want to find the shortest path from s to t in G. The length of this shortest path is called the "distance" from s to t.

A "feasible potential" γ in G is an assignment of numbers γ_v to the vertices v of V such that :
(1) $\gamma_s = 0$
(2) for every edge uv in E we have $\gamma_v - \gamma_u \leq b_{uv}$.



1) Why is it important to assume that G contains no negative cycle ?
2) Prove that if γ is a feasible potential in G, then γ_t is a lower bound on the distance from s to t.
3) Use the LP duality theorem to prove that there exists a feasible potential γ for which γ_t equals the distance from s to t.
Hint : Formulate the problem of finding the shortest path from s to t as an integer program. Prove that the LP relaxation has an integer solution. Construct the dual of this LP.

Solution sketch :
1) If G contains a negative cycle then there might be arbitrarily small paths from s to t, by taking the negative cycle arbitrarily many times on the way from s to t. Furthermore, if G contains a negative cycle then G does not have any feasible potential.
2) Consider a path $s, v_1, v_2, ..., v_k, t$. Its length is $b_{sv_1} + b_{v_1v_2} + ... + b_{v_kt} \geq (\gamma_{v_1} - \gamma_s) + (\gamma_{v_2} - \gamma_{v_1}) + ... + (\gamma_t - \gamma_{v_k}) = \gamma_t - \gamma_s = \gamma_t$.
3) The relaxed LP is :

minimize
. $\sum_{e \in E} x_e b_e$
subject to
. $\sum_{e \text{ enters } v} x_e - \sum_{e \text{ leaves } v} x_e = 1$ if $v = t$, -1 if $v = s$, 0 otherwise for every $v \in V$,
. $x_e \geq 0$ for every e in E.

Here x_e denotes how many times we take the edge e in the path from s to t. Suppose we have a feasible solution for this LP in which some x_e's are fractional. If a vertex is adjacent to a fractional edge, then it must have another fractional edge adjacent to it (since the right-hand-side of the vertex's constraint is an integer). Therefore we can find a cycle of fractional edges (where some of the edges in the cycle might point forwards and some might point backwards). We take this cycle, and for each forwards edge we add ϵ to x_e and for each backwards edge we add -ϵ. We either decrease or increase ϵ starting from 0, so as to decrease the objective function, until one of the x_e's becomes an integer. We have thus decreased the number of fractional x_e's ; we repeat until all x_e's are integers.
Furthermore, there exists an optimal solution in which each x_e is either 0 or 1, because if some $x_e \geq 2$ then the path from s to t goes over itself, so it can be simplified making it only shorter.

The dual LP is as follows :

maximize
- $\gamma_t - \gamma_s$

subject to
- $\gamma_v - \gamma_u \leq b_{uv}$ for every edge uv in E.

Without loss of generality we can set $\gamma_s = 0$, and we get exactly the feasible potential problem.