DISCRETE OPTIMIZATION ASSIGNMENT 11

EXERCISE 1

Prove the following lemma (known as "complementary slackness") using the duality theorem.
LEMMA
Let $x^* = (x_1^*, ..., x_n^*)$ be a feasible solution of the linear program

$$\begin{aligned} &\text{Maximize } c^T x \\ &\text{subject to : } Ax \leq b, x \geq 0; \end{aligned} \qquad (P)$$

and let $y^* = (y_1^*, ..., y_m^*)$ be a feasible solution of the dual linear program :

$$\begin{aligned} &\text{Minimize } b^T y \\ &\text{subject to : } A^T y \geq c, y \geq 0. \end{aligned} \qquad (D)$$

Then the following two conditions are equivalent :
I) x^* is an optimal solution for (P) and y^* is an optimal solution for (D).
II) For every $i, 1 \leq i \leq m$, if x^* does not satisfy the i-th constraint of (P) with equality then $y_i^* = 0$; and for every j, $1 \leq j \leq n$, if y^* does not satisfy the j-th constraint of (D) with equality then $x_j^* = 0$.
SOLUTION 1
We have

$$Ax^* \leq b \qquad (1)$$
$$A^T y^* \geq c. \qquad (2)$$

Equation (2) is equivalent to $y^{*T} A \geq c^T$. We left-multiply both sides of (1) by y^{*T} (this does not change the $'' \leq ''$ because y^* is nonnegative), and we get

$$y^{*T} A x^* \leq y^{*T} b.$$

Putting everything together, we get

$$c^T x^* \leq y^{*T} A x^* \leq y^{*T} b. \qquad (3)$$

This is weak duality, which says that the objective function of the dual is always at least the objective function of the primal.
The duality theorem says that x^* and y^* are optimal if and only if the objective functions are equal, i.e. $c^T x^* = y^{*T} b$. That happens if and only if both inequalities of (3) are equalities.
Let us look at the first inequality of (3). It can be rewritten as $x^*(y^{*T} A - c^T) \geq 0$. Here the left-hand-side is zero if and only if, for every position j, either $x_j^* = 0$ or the j-th entry of $y^* TA - c^T$ is zero. (This is exactly the second condition of II).
Similarly, the second inequality of (3) can be rewritten as $y^{*T}(b - Ax^*) \geq 0$. Here the left-hand-side is zero if and only if, for every position i, either $y_i^* = 0$ or the i-th entry of $b - Ax^*$ is zero. (This is exactly the first condition of II).

EXERCISE 2
Consider the following LP :

$$\text{Maximize } x_1 + x_2$$

$$\text{subject to :}\quad \begin{aligned} 2x_1 + x_2 &\le 6 \\ x_1 + 2x_2 &\le 8 \\ 3x_1 + 4x_2 &\le 22 \\ x_1 + 5x_2 &\le 23 \\ x_1, x_2 &\ge 0. \end{aligned}$$

Use complementary slackness to prove that $x^* = (4/3, 10/3)$ is an optimal solution.
SOLUTION 2

The vector x^* achieves objective function 14/3. We check that x^* satisfies only the first two constraints with equality. Therefore, these are the only constraints that we can use in the dual. Furthermore, both components of x^* are nonzero, and that means that in the dual we must exactly achieve the corresponding coefficients of x_1 and x_2 (namely, 1 and 1).

Knowing this, it is easy to find the solution by inspection : If we multiply the first constraint by 1/3 and the second constraint by 1/3 and we add them together, we obtain :

$$x_1 + x_2 \le 14/3.$$

This proves that x^* is optimal.