ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Bachelor semester 4-DO

Discrete Optimization last WEEK

EXERCISE 1
(This exercise shows a real-world application of graphs that can have negative lengths.)

We are trading currencies. We are given a set of possible currency exchange rates, where each each rate consists of a pair v, w of currencies, together with a rate r_{vw} that specifies the number of units of currency w that we can purchase for one unit of currency v.
a. Explain how to represent this information as a directed graph, so that finding the best way of converting currency s to currency t corresponds to finding the shortest path from s to t in the graph.
b. What does a negative cycle in the graph represent ?

Solution
a. Label each directed edge vw with $-log_{10} r_{vw}$. Then, if L is the length of the shortest path from s to t, then 10^{-L} is the amount of currency t that we get per unit of currency s.
b. A way to make money from the exchange agency.

EXERCISE 2
A directed graph $G = (V, E)$ is "acyclic" if it does not contain any directed cycle. A "topological sort" of a graph is an ordering of its vertices as $v_1, v_2, ..., v_n$ such that every edge goes from left to right.
a. Prove that G is acyclic if and only if it has a topological sort, and provide a linear-time algorithm that either finds a topological sort of G or says that G is cyclic.
(Hint : An acyclic graph must contain a vertex without incoming edges.)
b. Describe a linear-time algorithm that finds all shortest paths from a given vertex s in a given acyclic graph.

Solution
a. If G is cyclic then clearly it cannot have a topological sort. If G is acyclic then it must contain a vertex without incoming edges (for otherwise we could start at a vertex, and go on and on backwards until we close a cycle). Call this vertex v_1, remove it and its edges from G, and so on.
For the linear-time algorithm, we keep a set S of vertices that have no incoming edges. At each step we take a vertex v out of S, and we remove v and its edges, checking for each edge vw whether w should now enter S. If at any stage S becomes empty before the graph runs out of vertices, then we know that G contains a cycle.
b. Given the acyclic graph, first obtain a topological sort $v_1, v_2, ..., v_n$ of it. Then do a *single round* of Bellman-Ford in this order, meaning, $do\_vertex(v_1), do\_vertex(v2), ..., do\_vertex(v_n)$.

EXERCISE 3
Prove Helly's theorem in one dimension, meaning : If $L_1, L_2, ..., L_n$ is a family of intervals of real numbers such that every pair of intervals intersect, then all the intervals intersect (meaning, there exists a real number that is contained in all the intervals).

Solution
Take the rightmost left-endpoint of all the intervals. It must be contained in all the intervals.]