Bachelor semester 4-DO

DISCRETE OPTIMIZATION WEEK 2

EXERCISE 1

We are given a finite set $S \subseteq \mathbb{R}^2$ of points in the plane, and somebody claims that these points lie appoximately on a circle. How can this be verified ? Here is one possible approach : An *annulus* (ring) is the region between two concentric circles. The *smallest enclosing annulus* of S is the annulus of smallest area that contains S :

If S indeed lies on a circle, then this area is 0, but also if S is "almost" on a circle, the smallest enclosing annulus will form a (small) ring around that circle.

Show that the problem of computing the smallest enclosing annulus of $S \subseteq \mathbb{R}^2$ can be formulated as a linear program !

SOLUTION

Let $(a_1, b_1), ..., (a_n, b_n)$ denote the points in S. Create 4 unknown variables, x, y, u, v, where (x, y) denotes the center of the concentric circles, $u = r_1^2 - x^2 - y^2$, $v = r_2^2 - x^2 - y^2$ where r_1, r_2 are the radius of the smallest and largest circle, respectively.

For every point the following constraint must be satisfied :

$$r_1^2 \leq (x - a_i)^2 + (y - b_i)^2 \leq r_2^2 \tag{1}$$

which is equivalent to

$$r_1^2 - x^2 - y^2 \leq -2xa_i - 2yb_i + a_i^2 + b_i^2 \leq r_2^2 - x^2 - y^2. \tag{2}$$

Note that the area of the annulus is $\pi(u - v)$. Hence we can write the linear program as min $\pi(u - v)$ subject to

$$u \leq -2xa_i - 2yb_i + a_i^2 + b_i^2 \leq v \quad \forall i \in \{1, 2, ..., n\}. \tag{3}$$

Once a solution to the linear program is obtained, all we need to do is to calculate the radius of each circle. From the definitions, we get that $r_1 = \sqrt{u + x^2 + y^2}$ and $r_2 = \sqrt{v + x^2 + y^2}$. The center of the annulus is (x, y).

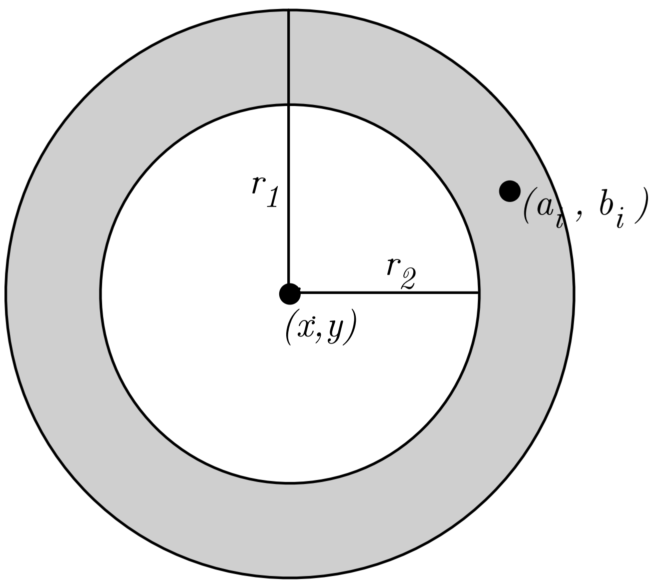

FIG. 1 –

EXERCISE 2

Postponed to the following week.

EXERCISE 3

A paper mill manufactures rolls of paper of a standard width 4 meters. But customers want to buy paper rolls of shorter width, and the mill has to cut such rolls from the 4 meters rolls.

Consider an order of

- 106 rolls of width 68 cm,
- 28 rolls of width 173 cm,
- 205 rolls of width 148 cm,
- 93 rolls of width 127 cm.

What is the smallest number of 4 m rolls that have to be cut in order to satisfy this order ? Write a linear program where the solution is the answer to this question.

SOLUTION

First, we write down all possibilities of cutting a 4 m paper roll into rolls of some the required widths. We only need to consider the possibilities where the wasted piece is shorter than 68 cm :

- P1 : 2 × 173
- P2 :173+ 148 + 68
- P3 : 173+127 + 68
- P4 : 173 + 3 × 68
- P5 : 2 × 148 + 68
- P6 : 148 + 127 + 68
- P7 : 148 + 3 × 68
- P8 : 127 + 4 × 68
- P9 : 5 × 68
- P10 : 3 × 127
- P11 : 2 × 127 + 2 × 68

For each possibility P_j on the list we introduce a variable $x_j \geq 0$ representing the number of rolls cut according to that possibility. We want to minimize the total number of rolls cut, in such a way that each customer is satisfied. The following linear program solves such problem.

min $\sum_{j=1}^{11} x_j$ subject to

$x_2 + x_3 + 3x_4 + x_5 + x_6 + 3x_7 + 4x_8 + 5x_9 + 2x_{11} \geq 106$

$2x_1 + x_2 + x_3 + x_4 \geq 28$

$x_2 + 2x_5 + x_6 + x_7 \geq 205$

$x_3 + 2x_6 + x_8 + 3x_{10} + 2x_{11} \geq 93$