DISCRETE OPTIMIZATION WEEK 4

EXERCISE 1

Let p_1, p_2, p_3, p_4 and p_5 be five points in $\mathbb{R}^d$, and let $q \in \mathbb{R}^d$ be the convex combination

$$q = \frac{2}{9}p_1 + \frac{4}{9}p_2 + \frac{1}{9}p_3 + \frac{1}{9}p_4 + \frac{1}{9}p_5$$

Find points r_1, r_2 and r_3 in $\mathbb{R}^d$ such that r_1 lies on the segment p_1p_2, r_2 lies on the segment r_1p_3, r_3 lies on the segment r_2p_4, and q lies on the segment r_3p_5.

$$\begin{aligned} q &= \frac{2}{9}p_1 + \frac{4}{9}p_2 + \frac{1}{9}p_3 + \frac{1}{9}p_4 + \frac{1}{9}p_5 \\ &= \frac{8}{9}\left(\frac{2}{8}p_1 + \frac{4}{8}p_2 + \frac{1}{8}p_3 + \frac{1}{8}p_4\right) + \frac{1}{9}p_5 \\ &= \frac{8}{9}\left(\frac{7}{8}\left(\frac{2}{7}p_1 + \frac{4}{7}p_2 + \frac{1}{7}p_3\right) + \frac{1}{8}p_4\right) + \frac{1}{9}p_5 \\ &= \frac{8}{9}\left(\frac{7}{8}\left(\frac{6}{7}\left(\frac{2}{6}p_1 + \frac{4}{6}p_2\right) + \frac{1}{7}p_3\right) + \frac{1}{8}p_4\right) + \frac{1}{9}p_5 \end{aligned}$$

So, $r_1 = \frac{1}{3}p_1 + \frac{2}{3}p_2$, $r_2 = \frac{6}{7}r_1 + \frac{1}{7}p_3$, $r_3 = \frac{7}{8}r_2 + \frac{1}{8}p_4$, and $q = \frac{8}{9}r_3 + \frac{1}{9}p_5$.

EXERCISE 2

Let $X = [0,1]^3$ be the unit cube in $\mathbb{R}^3$, and suppose we have a linear program whose set of feasible solutions is X. Find an objective function for which :

a) the unique optimal solution is the vertex $(1,0,0)$;

$$f(x_1, x_2, x_3) = x_1 - x_2 - x_3$$

b) the set of optimal solutions is the edge $(1,0,0) - (1,1,0)$;

$$f(x_1, x_2, x_3) = x_1 - x_3$$

c) the set of optimal solutions is the face with vertices $(1,0,0), (1,1,0), (1,0,1), (1,1,1)$;

$$f(x_1, x_2, x_3) = x_1$$

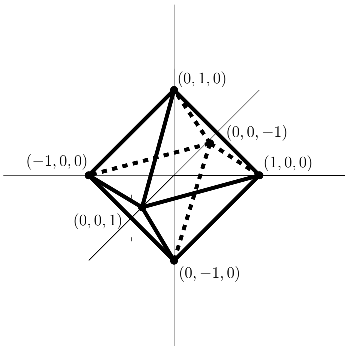
d) the set of optimal solutions is all of X.

$$f(x_1, x_2, x_3) = 0$$

EXERCISE 3

a) The n-dimensional crosspolytope is the set $C_n = \{x \in \mathbb{R}^n : |x_1| + |x_2| + ... + |x_n| \leq 1\}$. The following picture shows C_3 :
Express C_n as the solution set of a linear system of inequalities (meaning, a system of the form $Ax \leq b$).
The set C_n is determined as the solution of the following system of inequalities :

$$(-1)^{i_1}x_1 + (-1)^{i_2}x_2 + \ldots + (-1)^{i_n}x_n \leq 1, \quad \text{for all } (i_1, i_2, \ldots, i_n) \in \{0,1\}^n$$



b) Let $D = \{x \in \mathbb{R}^3 : |2x_1 - x_2 + 3x_3 + 1| + |x_2 + 2x_3 - 2| + |5x_1 - 3x_3| \leq 10\}$. Express D as the solution set of a linear system of inequalities.

Similarly as in the previous exercise we have :

$$(-1)^{i_1}(2x_1 - x_2 + 3x_3 + 1) + (-1)^{i_2}(x_2 + 2x_3 - 2) + (-1)^{i_3}(5x_1 - 3x_3) \leq 10, \quad \text{for all } (i_1, i_2, i_3) \in \{0,1\}^3$$

By simplifying the equalities we get :

$$\begin{aligned}
-7x_1 - 2x_3 &\leq 9 \\
3x_1 - 8x_3 &\leq 9 \\
3x_1 + 2x_2 - 4x_3 &\leq 13 \\
-7x_1 + 2x_2 + 2x_3 &\leq 13 \\
-3x_1 - 2x_2 + 4x_3 &\leq 7 \\
7x_1 - 2x_2 - 2x_3 &\leq 7 \\
7x_1 + 2x_3 &\leq 11 \\
-3x_1 + 8x_3 &\leq 11
\end{aligned}$$

EXERCISE 4

Convert the following linear program to equational form :

Minimize $2x_1 - 3x_2$

Subject to :

$$\begin{aligned}
x_1 + 5x_2 &\leq 10 \\
-x_1 + 2x_2 &\geq -5 \\
x_1 &\geq 0
\end{aligned}$$

First, we introduce two new variables y, z and substitute x_2 for $y - z$. Then we introduce two slack variables s_1 and s_2. Finally, we multiply objective function by -1 in order to get a maximizing linear program.

Maximize $-2x_1 + 3y - 3z$

Subject to :

$$\begin{aligned}
x_1 + 5y - 5z + s_1 &= 10 \\
-x_1 + 2y - 2z - s_2 &= -5 \\
x_1, y, z, s_1, s_2 &\geq 0
\end{aligned}$$