DISCRETE OPTIMIZATION WEEK 6

EXERCISE 1

We showed in class that if a linear program in *equational form* is feasible and bounded, then it has an optimal solution which is a vertex of the polyhedron of feasible solutions. Show that this is not always true for general linear programs. Namely, construct a linear program that is feasible and bounded but whose set of optimal solutions does not contain any vertex.

Solution : For example, maximize $x_1 + x_2$ subject to $x_1 + x_2 \leq 1$.

EXERCISE 2

Consider the following simplex tableau. Specify the entering variable and the leaving variable according to the following pivot rules :

a) Largest coefficient
b) Largest increase
c) **optional**
Steepest edge
d) Bland's rule

$$\begin{array}{rcl} x_1 & = & 4 + 2x_4 - x_5 - 6x_6 \\ x_2 & = & 3 - 3x_4 - 2x_5 - 3x_6 \\ x_3 & = & 2 - x_4 + 5x_5 - 3x_6 \\ \hline z & = & 3 - 5x_4 + x_5 + 2x_6 \end{array}$$

Solution :

a) The largest coefficient has x_6. For $x_6 = \frac{2}{3}$, x_1 or x_3 leaves the basis. The choice is arbitrary, but below we stick to the latter.

$$\begin{array}{rcl} x_1 & = & 2x_3 + 4x_4 - 11x_5 \\ x_2 & = & 1 + x_3 - 2x_4 - 7x_5 \\ x_6 & = & \frac{2}{3} - \frac{x_3}{3} - \frac{x_4}{3} + \frac{5}{3}x_5 \\ \hline z & = & \frac{13}{3} - \frac{2}{3}x_3 - \frac{17}{3}x_4 + \frac{13}{3}x_5 \end{array}$$

b) The largest increase is achieved by increasing x_5 to $\frac{3}{2}$. Thus, the leaving variable is x_2. The return value of the objective function denoted by z in this case increases by $\frac{3}{2}$. Note that by increasing x_6 we gain only $\frac{4}{3}$.

$$\begin{array}{rcl} x_1 & = & \frac{5}{2} + x_2 + \frac{7}{2}x_4 - \frac{9}{2}x_6 \\ x_5 & = & \frac{3}{2} - x_2 - \frac{3}{2}x_4 - \frac{3}{2}x_6 \\ x_3 & = & \frac{19}{2} - 5x_2 - \frac{17}{2}x_4 - \frac{21}{2}x_6 \\ \hline z & = & \frac{9}{2} - x_2 - \frac{13}{2}x_4 + \frac{1}{2}x_6 \end{array}$$

c) Steepest edge. The vector c corresponding to the objective function is $(0, 0, 0, -5, 1, 2)$. We want to maximimze

$$s = \frac{< c, (\mathbf{x}_{\text{new}} - \mathbf{x}_{\text{old}}) >}{||\mathbf{x}_{\text{new}} - \mathbf{x}_{\text{old}}||}$$

Where $\mathbf{x}_{\text{old}} = (4, 3, 2, 0, 0, 0)$. In case x_5 is chosen to be the variable for entering the basis $\mathbf{x}_{\text{new}} = (\frac{5}{2}, 0, \frac{19}{2}, 0, \frac{3}{2}, 0)$. In case x_6 is chosen to be the variable for entering the basis we have $\mathbf{x}_{\text{new}} = (0, 1, 0, 0, 0, \frac{2}{3})$. The straightforward calculation reveals that $s = \frac{3}{\sqrt{279}}$ in case of x_5 and $s = \frac{4}{\sqrt{210}}$ in case of x_6. Hence, according to this rule we go for x_6, and the leaving variable can be again chosen arbitrarily between x_1 and x_3. See a) for the tableau.

d) For Bland's rule the entering variable is x_5 and the leaving variable is x_2. See b) for the tableau.

EXERCISE 3

Let P be the following linear program in equational form :

Maximize $\quad -x_1 + 3x_2 - x_3 + 3x_4$

Subject to :

$$\begin{aligned} 3x_1 + 2x_2 - 7x_3 + x_4 &= -1 \\ 2x_1 + x_2 + x_3 + x_4 &= 5 \\ x_1 - 3x_2 - 3x_3 &= 10 \end{aligned}$$

Which auxiliary linear program do we have to solve in order to find a basic feasible solution for P ?

Solution :

Maximize $\quad -x_5 - x_6 - x_7$

Subject to :

$$\begin{aligned} -3x_1 - 2x_2 + 7x_3 - x_4 + x_5 &= 1 \\ 2x_1 + x_2 + x_3 + x_4 + x_6 &= 5 \\ x_1 - 3x_2 - 3x_3 + x_7 &= 10 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

Note that we multiply the first equation by -1 to get a non-negative right-hand side.