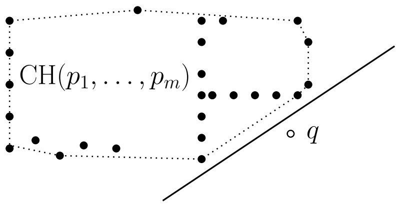
DISCRETE OPTIMIZATION WEEK 9

EXERCISE 1

Let $\mathbf{p_1}, \ldots, \mathbf{p_m}, \mathbf{q}$ be points in $\mathbb{R}^n$. Prove that if $\mathbf{q} \notin \mathrm{CH}(\mathbf{p_1}, ..., \mathbf{p_m})$ then there exists a hyperplane that separates $\mathbf{q}$ from $\mathbf{p_1}, \ldots, \mathbf{p_m}$.



Hint : Write a system S of linear equalities and inequalities such that S has a solution if and only if $\mathbf{q} \in \mathrm{CH}(\mathbf{p_1}, ..., \mathbf{p_m})$. Apply Farkas's lemma to S, and interpret the result geometrically.

Solution :

The point $\mathbf{q}$ belongs to $\mathrm{CH}(\mathbf{p_1}, ..., \mathbf{p_m})$ if and only if it is a convex combination of these points, meaning, if and only if there exist nonnegative numbers $x_1, \ldots, x_m$ such that $x_1\mathbf{p_1} + \ldots + x_m\mathbf{p_m} = \mathbf{q}$, $x_1 + ... + x_m = 1$. Letting

$$\mathbf{A} = \begin{pmatrix} \mathbf{p_1} & \cdots & \mathbf{p_m} \\ 1 & \ldots & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} \mathbf{q} \\ 1 \end{pmatrix}, \ \mathbf{x} = (x_1, \ldots, x_m)$$

This is equivalent to saying that the system $\mathbf{Ax} = \mathbf{b}$ has a nonnegative solution. If it has no solution, then by Farkas's lemma there exists a vector $\mathbf{y} = (y_1, ..., y_n, y_{n+1})$ such that $\mathbf{y}^T\mathbf{A} \geq 0, \mathbf{y}^T\mathbf{b} < 0$. But this just means that the points $\mathbf{p_1}, \ldots, \mathbf{p_m}$ are on one side of the hyperplane H and $\mathbf{q}$ is on the other side of H, for $H = \{\mathbf{z} \in \mathbb{R}^n | \ y_1z_1 + ... + y_nz_n + y_{n+1} = 0\}$. So this is the hyperplane we were looking for. QED